Computational Finance
Questions

Tutorial 1

Exercise 1

1. Let x and y be two dependent random variables, and let α and β be real numbers. Prove that

$$\mathrm{var}(\alpha x + \beta y) = \alpha^2 \mathrm{var}(x) + 2\alpha\beta \mathrm{cov}(x, y) + \beta^2 \mathrm{var}(y).$$

2. Suppose that there are two stocks. Let x and y denote the random values of the first and second stock, respectively, after one year. Furthermore, we know that $\mathrm{std}(x) = 0.20$, $\mathrm{std}(y) = 0.18$, and $\mathrm{cov}(x, y) = 0.01$. A *portfolio* is composed out of $\alpha = 2$ units of stock 1 and $\beta = 3$ units of stock 2. Calculate the variance of the *portfolio value* in one year, that is, $\mathrm{var}(\alpha x + \beta y)$.

Exercise 2

Find the mean and the variance of a random variable described by the probability density function

$$p(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Exercise 3

Write the second order Taylor series expansion of

1. $f(x) = e^x$, around $x = 1$.

1. Let x and y be two dependent random variables, and let α and β be real numbers. Prove that

$$\mathrm{var}(\alpha x + \beta y) = \alpha^2 \mathrm{var}(x) + 2\alpha\beta \mathrm{cov}(x, y) + \beta^2 \mathrm{var}(y).$$

$var(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2$ $cov(x,y) = E[(X - E[X]) \cdot (Y - E[Y])]$

$E[\alpha X + \beta Y] = \alpha \cdot E[X] + \beta \cdot E[Y]$

$$var(\alpha X + \beta Y) = E[(\alpha X + \beta Y - E[\alpha X + \beta Y])^2] =$$

$$= E[(\alpha X + \beta Y - \alpha E[X] - \beta E[Y])^2]$$

$$= E[(\alpha \cdot (X - E[X]) + \beta \cdot (Y - E[Y]))^2]$$

$$= E[(\alpha \cdot (X - E[X]))^2 + 2 \cdot \alpha \cdot \beta \cdot (X - E[X]) \cdot (Y - E[Y]) + (\beta \cdot (Y - E[Y]))^2]$$

$$= E[\alpha^2 (X - E[X])^2] + E[2\alpha\beta \cdot (X - E[X]) \cdot (Y - E[Y])] + E[\beta^2 \cdot (Y - E[Y])^2]$$

$$= \alpha^2 \cdot var(x) + 2\alpha\beta \cdot cov(x, y) + \beta^2 \cdot var(y)$$

✓

2. Suppose that there are two stocks. Let x and y denote the random values of the first and second stock, respectively, after one year. Furthermore, we know that $\mathrm{std}(x) = 0.20$, $\mathrm{std}(y) = 0.18$, and $\mathrm{cov}(x, y) = 0.01$. A *portfolio* is composed out of $\alpha = 2$ units of stock 1 and $\beta = 3$ units of stock 2. Calculate the variance of the *portfolio value* in one year, that is, $\mathrm{var}(\alpha x + \beta y)$.

$std(x) = \sqrt{var(x)}$ $cov(x,y) = E[(X - E[X]) \cdot (Y - E[Y])]$

$$var(2x + 3y) = 4 \cdot var(x) + 12 \cdot cov(x,y) + 9 \cdot var(y)$$

$$= 4 \cdot 0{,}04 + 12 \cdot 0{,}01 + 9 \cdot 0{,}0324$$

$$= 0{,}5716$$

Find the mean and the variance of a random variable described by the probability density function

$$p(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$\frac{12}{3} - \frac{8}{3} = \frac{4}{3}$

$$E(X) = \int_{-\infty}^{+\infty} p(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 2x - x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 + \left[x^2 - \frac{1}{3}x^3\right]_1^2 = \frac{1}{3} + \frac{2}{3} = 1$$

$1 - \frac{1}{3} = \frac{2}{3}$

$$var(X) = E(x^2) - E(X)^2 = -1 + \int_0^1 x^3\,dx + \int_1^2 2x^2 - x^3\,dx = \left[\frac{1}{4}x^4\right]_0^1 + \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_1^2 - 1 = \frac{3}{12} + \frac{11}{12} - 1 = \frac{1}{6}$$

$\frac{16}{3} - \frac{12}{3} = \frac{4}{3} = \frac{16}{12}$ $= \frac{11}{12}$

$\frac{2}{3} - \frac{1}{4} = \frac{8}{12} - \frac{3}{12} = \frac{5}{12}$

Simple numerical errors were my primary mistake!

Write the second order Taylor series expansion of

1. $f(x) = e^x$, around $x = 1$.
2. $f(x) = e^{x^2}$, around $x = 1$.
3. $f(x_1, x_2) = e^{x_1 x_2}$, around $x_1 = x_2 = 0$.

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

2^{nd}-order: $g(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)\cdot(x-a)^2}{2!}$

a) at 1

$f(x) = e^x \quad = e$

$f'(x) = e^x \quad = e \implies g(x) = e^1 + e^1\cdot(x-1) + e^1 \cdot \frac{(x^2-2x+1)}{2} = e^1\left(1+x-1+\frac{x^2}{2}-x+\frac{1}{2}\right)$

$f''(x) = e^x \quad = e$

$= e^1\cdot\left(\frac{x^2}{2}+\frac{1}{2}\right)$

b) at 1

$(f(g(x)))' = f'(g(x))\cdot g'(x)$

$(f(x)\cdot g(x))' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

$f(x) = e^{x^2} \quad = e$

$f'(x) = e^{x^2}\cdot 2x \quad = 2e$

$f''(x) = (e^{x^2})'\cdot 2x + e^{x^2}\cdot 2 = 2e^{x^2} + 2x\cdot(2x\cdot e^{x^2}) = 2e^{x^2} + 4x^2\cdot e^{x^2} \quad = 6e$

$g(x) = e + 2e(x-1) + 3e\cdot(x^2-2x+1) = e + 2ex - 2e + 3ex^2 - 6ex + 3e$

$= 3ex^2 - 4ex + 2e$

c) at (0,0)

$f(x,y) = e^{x\cdot y} \quad = 1$

$f_x(x,y) = y\cdot e^{xy} \quad = 0$

$f_y(x,y) = x\cdot e^{xy} \quad = 0$

$f_{xx}(x,y) = y^2\cdot e^{xy} \quad = 0$

$f_{yy}(x,y) = x^2\cdot e^{xy} \quad = 0$

$f_{xy}(x,y) = (y)'\cdot e^{xy} + y\cdot(e^{xy})' = e^{xy} + y\cdot x\cdot e^{xy} = e^{xy}\cdot(1+xy) \quad = 1$

$g(x) = f(0,0) + f_x(0,0)\cdot(x) + f_y(0,0)\cdot(y) + \frac{f_{xx}(0,0)\cdot(x)^2}{2} + f_{xy}(0,0)(x)\cdot(y) + \frac{f_{yy}(0,0)\cdot(y)^2}{2}$

$g(x) = x\cdot y + 1$

Lagrange multipliers:

$\boxed{\text{risk}} = \text{variance}$

$\boxed{\text{budget}} = 10^6$

$\boxed{\text{target}} = 1{,}1$

$w_i :=$ % of wealth in asset i

$\sum_i w_i = 1$ → first constraint