Lecture 2: Quantum Circuits

Outline

- State and gates of multiple qubits
- Quantum circuits diagrams
- Entangled states and generalised Born rule
- Universality and efficiency of quantum algorithms

Intended Learning Outcomes

- Understanding and applying the matrix and quantum circuit representation of quantum gates.
- Remembering the definition of entanglement and the generalised Born rule for partial quantum measurement.
- Analysing the efficiency of quantum algorithms.

Why this matters

- Quantum circuit is the language in which quantum algorithms are described.
- Knowing the foundations of this chapter will allow you to understand the description of any quantum algorithm, including latest research.

1 Manipulating states of multiple qubits

1.1 States of multiple qubits and their unitary transformations

- State of n qubits

$$|\psi\rangle = \sum_{x=0}^{2^n-1} \alpha_x |x\rangle_n \ , \quad \langle\psi|\psi\rangle = \sum_{x=0}^{2^n-1} |\alpha_x|^2 = 1 \, .$$

  $\alpha_x \in \mathbb{C}$ and $|\psi\rangle \in \mathbb{C}^{2^n}$. $|\alpha_x|^2$ = probability of x, and $|\psi\rangle \equiv \mathrm{e}^{i\varphi} |\psi\rangle$.

- Since $x = \sum_{j=0}^{n-1} x_j 2^j$,

$$|\psi\rangle = \sum_{x_{n-1}=0}^{1} \sum_{x_{n-2}=0}^{1} \cdots \sum_{x_0=0}^{1} \alpha_{x_{n-1}x_{n-2}\cdots x_0} |x_{n-1}\rangle |x_{n-2}\rangle \cdots |x_0\rangle \ ,$$

  $\alpha_x \equiv \alpha_{x_{n-1}\cdots x_1 x_0}$.

- Linear operations preserve norm:

$$\langle\psi| U^\dagger U |\psi\rangle = \langle\psi|\psi\rangle = 1 \Rightarrow U^\dagger U = \mathbf{1}$$

  U: $2^n \times 2^n$ unitary matrix, reversible.

- Examples

  - Reversible classical gates C (permutations) are unitary, act linearly

$$\begin{aligned} &|\psi\rangle = \alpha_0 |0\rangle_2 + \alpha_1 |1\rangle_2 + \alpha_2 |2\rangle_2 + \alpha_3 |3\rangle_2 \ , \\ &C|0\rangle_2 = |1\rangle_2 \ , \quad C|1\rangle_2 = |2\rangle_2 \ , \quad C|2\rangle_2 = |3\rangle_2 \ , \quad C|3\rangle_2 = |0\rangle_2 \ , \\ &\Rightarrow |\phi\rangle = C|\psi\rangle = \\ &\Rightarrow \langle\phi|\phi\rangle = \end{aligned}$$

  - CNOT C_{20} on $|\psi\rangle$ of 3 qubits:

$$C_{20}|\psi\rangle = \sum_{x_2,x_1,x_0} \alpha_{x_2x_1x_0} |x_2\rangle |x_1\rangle |x_0 \oplus x_2\rangle \ ,$$

  - The tensor product of unitary gates is unitary

$$U = H \otimes \mathbf{1}_2 \otimes X = H_2 X_0 \, , \quad U|\psi\rangle = \sum_{x_2,x_1,x_0} \alpha_{x_2x_1x_0} (H|x_2\rangle) |x_1\rangle (X|x_0\rangle)$$

1.2 Quantum circuits with multiple qubits

- n wires or registers for n qubits, label bottom to top.
- Gates as boxes wires act on

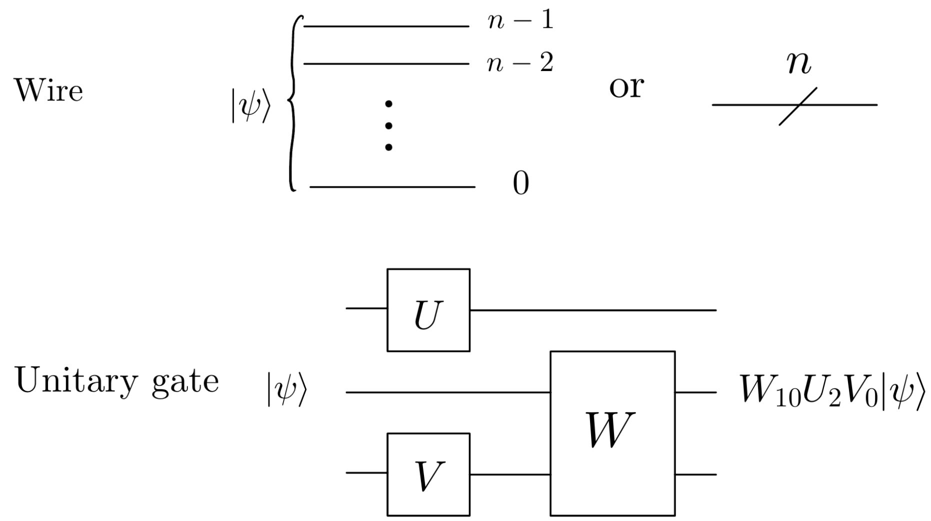

Figure 1

- Controlled-U (write C^U)

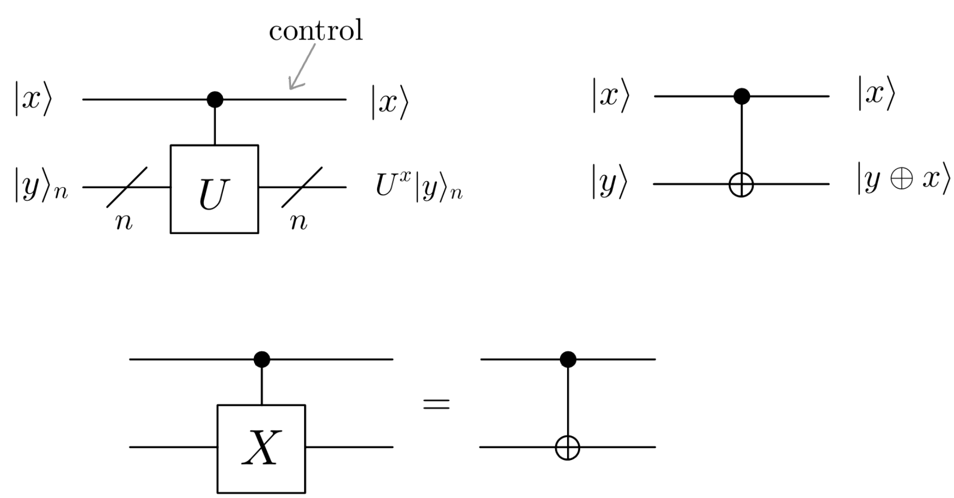

Figure 2

- $U = X$ is CNOT and has special notation

$$|x\rangle\, X^x\, |y\rangle = |x\rangle\, |y \oplus x\rangle$$

- Note: define action on basis state $|x\rangle\, |y\rangle$, extend on any inputs by linearity.

1.3 Entangled states

- n qubit state $|\psi\rangle$ is separable if $|\psi\rangle = |\psi_{n-1}\rangle \otimes |\psi_{n-2}\rangle \otimes \cdots \otimes |\psi_0\rangle$, $|\psi_i\rangle \in \mathbb{C}^2$. Otherwise, entangled.

- Example: Bell state (a.k.a. EPR pair after Einstein, Podolsky, Rosen)

$$|\psi_{00}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B)$$

- Is $|\psi\rangle = \frac{1}{2}(|00\rangle + |10\rangle - |01\rangle - |11\rangle)$ entangled? No:

  $H|0\rangle \otimes H|1\rangle =$

- This circuit prepares $|\psi_{00}\rangle$

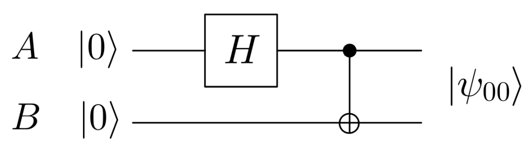

Figure 3

$|0\rangle |0\rangle \mapsto$

1.4 Generalised Born rule

- Measurement gates are non-reversible.
- If $|\psi\rangle = \sum_{x=0}^{2^n-1} \alpha_x |x\rangle_n$, measuring all qubits projects onto $|x\rangle_n$ and we readout x with probability $|\alpha_x|^2$.

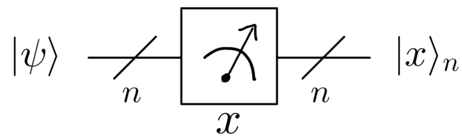

Figure 4

- Partial measurement, e.g. top one

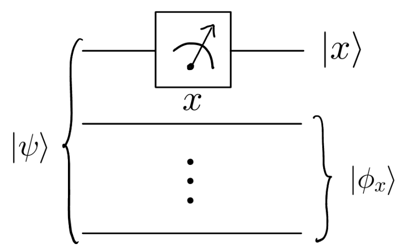

Figure 5

- We can expose top qubit as

$$\begin{aligned}|\psi\rangle &= \sum_{x_{n-1}} \sum_{x_{n-2},\cdots,x_0} \alpha_{x_{n-1}x_{n-2}\cdots x_0} |x_{n-1}x_{n-2}\cdots x_0\rangle \\ &= \\ &\equiv |0\rangle|\phi_0\rangle + |1\rangle|\phi_1\rangle\end{aligned}$$

- If measure $x \in \{0,1\}$ state is $|x\rangle|\phi_x\rangle$. Need to normalise! Generalised Born rule:

$$|\psi\rangle \mapsto \begin{cases} |0\rangle \frac{|\phi_0\rangle}{\| |\phi_0\rangle \|} & \text{with prob } \| |\phi_0\rangle \|^2 \\ |1\rangle \frac{|\phi_1\rangle}{\| |\phi_1\rangle \|} & \text{with prob } \| |\phi_1\rangle \|^2 \end{cases}$$

- $\| |\phi_x\rangle \|^2$ for probability mass of events compatible with measurement outcome:

$$\begin{aligned}\| |\phi_x\rangle \|^2 &= \sum_{x_{n-2},\ldots,x_0} \sum_{y_{n-2},\ldots,y_0} \overline{\alpha}_{xx_{n-2}\cdots x_0}\alpha_{xy_{n-2}\cdots y_0} \langle x_{n-2}|y_{n-2}\rangle \cdots \langle x_0|y_0\rangle \\ &= \sum_{x_{n-2},\ldots,x_0} |\alpha_{xx_{n-2}\cdots x_0}|^2\end{aligned}$$

- Example: Bell state of 2 qubits

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\phi_0\rangle =$$

$$|\phi_1\rangle =$$

Measurement leftmost qubit gives 0 with probability $\| \, |\phi_0\rangle \, \|^2 = \langle\phi_0|\phi_0\rangle = \frac{1}{2}$, and state is

$$|0\rangle \frac{|\phi_0\rangle}{\| \, |\phi_0\rangle \, \|} =$$

- Note: measuring m qubits same as m sequential 1 qubit measurements (see exercise)

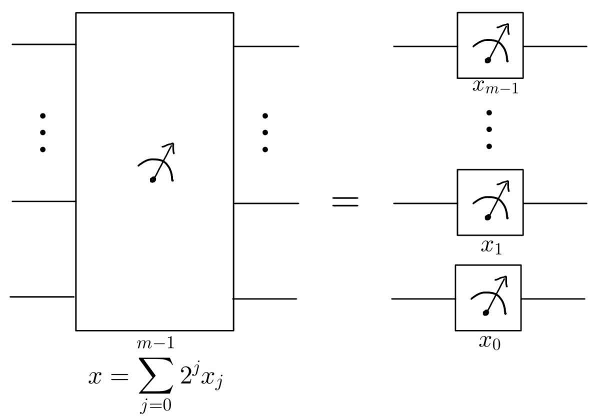

Figure 6

1.5 Universality and efficient quantum algorithms

- A quantum computer that can implement approximately any unitary on n qubits is called universal.
  - Elementary gate set $\mathcal{G}$ s.t. for each U there is $S_1, \ldots, S_k \in \mathcal{G}$ such that $\|U - S_1 \cdots S_k\| \leq \epsilon$
  - $\|A\|$: operator norm of A, $\max_{\psi \neq 0} \frac{\|A|\psi\rangle\|}{\||\psi\rangle\|}$.
- Single qubit gates and CNOT between any pair of qubits is universal (see exercise on implementing Toffoli).
- Quantum algorithms associate to input bits a quantum circuit – sequence of quantum gates. Measurement computes output

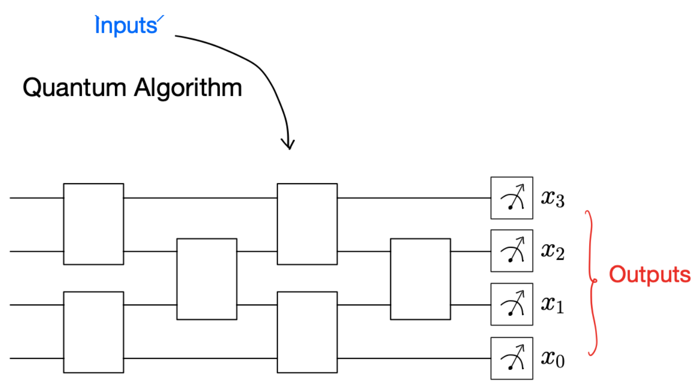

Figure 7: A quantum algorithm has input a bit string describing a computational problem and outputs a bit string corresponding to the output of the computational problem after measurement in a quantum circuit.

- **Runtime**: number of elementary gates. Efficient if $\mathcal{O}(p(n))$, n: input size.
- How to simulate a quantum circuit $U_\ell \cdots U_2 U_1$ on a classical computer?

$$\langle y| U_\ell \cdots U_2 U_1 |0^n\rangle = \sum_{x^{(\ell-1)} \ldots x^{(1)}} \langle y| U_\ell |x^{(\ell-1)}\rangle \cdots \langle x^{(1)}| U_1 |0^n\rangle$$

  $\langle x| U_i |y\rangle$ efficient since U_i acts on 1 or 2 qubits

$$\langle x| U_i |y\rangle = \langle x_2 x_1| U_i |y_2 y_1\rangle \prod_{k=3}^{n} \delta_{x_k, y_k} .$$

  If ℓ is poly n, memory poly n but time is exponential in n.

- We believe there is no efficient classical simulation algorithm of general quantum circuits.

Summary

In this chapter, we learnt that:

- States of n qubits are superpositions of classical bit strings and are described by vectors of length 2^n, normalised so that each entry squared is a probability of a measurement outcome.
- Quantum circuits give a convenient way to describe quantum a sequence of quantum gates.
- The most important unitary gates are the Pauli matrices, the Hadamard gate, and the CNOT gate.
- Measurement gates are irreversible. The generalised Born rule dictates the outcome of a measurement.
- Quantum algorithms are sequence of gates, efficient if polynomial in input size.