Reinforcement Learning

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Autumn Term 2025

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Reinforcement Learning
Lecture 1: Markov Decision Processes

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Outline

1. Introduction
2. Markov Decision Processes
3. Policies and Returns
4. Value Functions
5. Bellman Equations
6. Optimal Policies and Bellman Optimality

How does reinforcement learning actually work?

Intro: Why RL is amazing and will change the world

⬇️

Lecture 1: How does RL actually work?

Prerequisites

- Linear algebra, probability theory, calculus

Learning Outcomes

- Understand what makes RL different from other ML
- Master the mathematical framework: MDPs
- Learn the core concepts: policies, value functions, Bellman equations

What mathematical tools do we need for RL?

Probability Essentials

- **Conditional probability**: $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- **Chain rule**: $P(A \cap B) = P(A|B)P(B)$

Expectation

- **Definition**: $\mathbb{E}[X] = \sum_x x \cdot P(X = x)$
- **Linearity**: $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$
- **Conditional**: $\mathbb{E}[X|Y] = \sum_x x \cdot P(X = x|Y)$
- **Law of total expectation**: $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$

How does RL fit with other types of machine learning?

Paradigm	Learning Signal	Goal	Examples
Supervised	Labeled examples (x, y) pairs	Predict labels $y = f(x)$	Image classification Language translation
Unsupervised	Unlabeled data x only	Find patterns Density, structure	Clustering Dimensionality reduction
Reinforcement	Rewards (s, a, R, s')	Maximize return Optimal actions	Game playing Robot control

What makes RL uniquely challenging?

Key RL Challenges

- **Delayed consequences**: Actions have long-term effects
- **Exploration vs exploitation**: Try new things vs use what works
- **Credit assignment**: Which actions led to rewards?
- **Non-stationary**: Your actions change the data distribution

Why These Don't Exist in Supervised Learning

- **Immediate feedback**: Each (x, y) pair gives direct feedback
- **Fixed dataset**: No exploration needed, all data available
- **Clear attribution**: Each input directly maps to output
- **Stationary**: Data distribution doesn't change during training

How do we formalize sequential decision making problems?

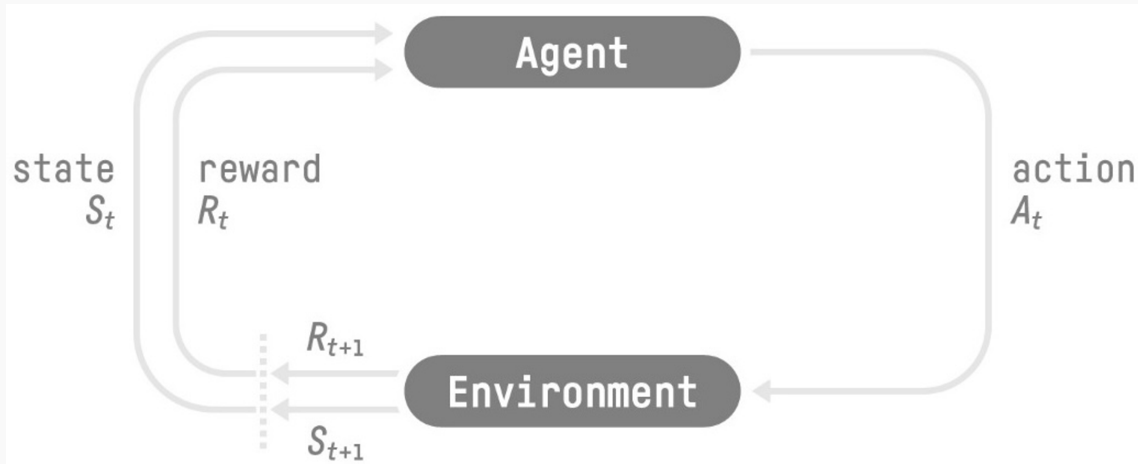
Definition (Markov Decision Process (MDP))
A model for sequential decision making when outcomes are uncertain.

Formally defined by the tuple $(\mathcal{S}, \mathcal{A}, P, R)$:

- $\mathcal{S}$: Set of **states** - all possible situations
- $\mathcal{A}$: Set of **actions** - all possible decisions
- $P(s'|s, a)$: **Transition probabilities** - how actions change states
- $R(s, a, s')$: **Reward function** - immediate feedback for transitions

If you can specify $(\mathcal{S}, \mathcal{A}, P, R)$, you can apply RL!

What does the agent-environment interaction look like?



The RL Loop:

1. Agent observes **state** s_t
2. Agent takes **action** a_t
3. Environment gives **reward** r_{t+1}
4. Environment transitions to new **state** s_{t+1}
5. Repeat...

Goal: Learn to maximize **cumulative reward**

Do tasks have natural start and end points?

Definition (Episode)
A complete sequence of interaction from start to terminal state:

$$S_0, A_0, R_1, S_1, A_1, R_2, S_2, \ldots, S_T$$

where S_T is a terminal state.

Types of MDPs

- **Episodic**: Has clear start/end states (games, tasks)
  - Chess game, robot reaching goal, Atari game
- **Continuing**: No terminal states (ongoing processes)
  - Stock trading, server management, autonomous driving

Why this matters: Affects how we define returns and value functions

What do states, actions, and rewards look like in practice?

Atari Breakout

- States: ?
- Actions: ?
- Rewards: ?

Robot Manipulation

- States: ?
- Actions: ?
- Rewards: ?

Autonomous Driving

- States: ?
- Actions: ?
- Rewards: ?

Recommendation Systems

- States: ?
- Actions: ?
- Rewards: ?

What do states, actions, and rewards look like in practice?

Atari Breakout

- **States**: Pixel observations
- **Actions**: Left, Right, Fire
- **Rewards**: +1 for brick hit

Robot Manipulation

- **States**: ?
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- **States**: ?
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Autonomous Driving

- **States**: Camera, lidar, GPS
- **Actions**: Steering, throttle, brake
- **Rewards**: Safe, efficient driving

Recommendation Systems

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Robot Manipulation

- **States**: Joint angles, object poses
- **Actions**: Joint torques/velocities
- **Rewards**: Task completion

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Recommendation Systems

- **States**: User history, context
- **Actions**: Recommend items
- **Rewards**: Clicks, purchases, ratings

Why start with finite state and action spaces?

Definition (Finite MDP)
An MDP where both state and action sets are finite:

- $|\mathcal{S}| = n < \infty$ (finite number of states)
- $|\mathcal{A}| = m < \infty$ (finite number of actions)

Why Start With Finite MDPs?

- **Mathematical tractability**: Can represent everything as matrices/tables
- **Exact solutions**: Can find optimal policies exactly
- **Clear intuition**: Easy to visualize and understand
- **Foundation for continuous**: Continuous methods often discretize or generalize these ideas

How do we model uncertainty in state transitions?

Definition (Dynamics/Transition Function)

$$P(s'|s,a) = \Pr[S_{t+1} = s'|S_t = s, A_t = a]$$

The probability of transitioning to state s' given current state s and action a.

Key Properties

- **Probability distribution**: $\sum_{s' \in \mathcal{S}} P(s'|s,a) = 1$ for all s, a
- **Stationary**: Transition probabilities don't change over time
- **Markovian**: Next state depends only on current state and action

Deterministic Example:

- GridWorld: $P(s'|s, \text{Right}) = 1$ if s' is right of s

Stochastic Example:

- Sticky GridWorld: $P(s'|s, \text{Right}) = 0.9$ if s' is right, $P(s|s, \text{Right}) = 0.1$

Why is the Markov property so important?

Definition (Markov Property)
The future is independent of the past given the present:

$$P(s_{t+1}|s_t, a_t, s_{t-1}, a_{t-1}, \ldots, s_0, a_0) = P(s_{t+1}|s_t, a_t)$$

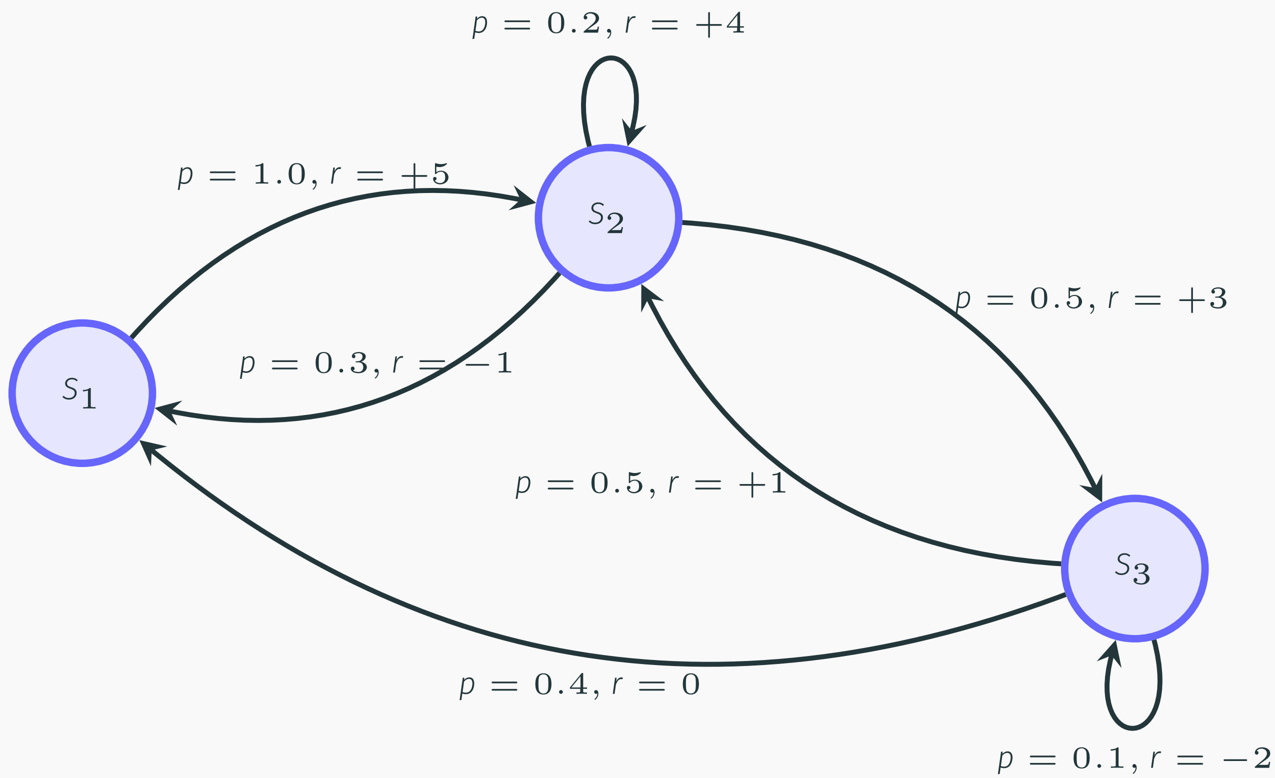
What This Means

- The current state contains all information needed to predict the future
- We don't need to remember the entire history
- Makes the problem mathematically tractable

Markovian vs Non-Markovian Examples

- **Markovian**: Chess position, robot joint angles + velocities
- **Non-Markovian**: Single Atari frame (no velocity), stock prices, poker (hidden cards)

How can we visualize MDP structure?



State Transition Diagram:

- Nodes = states, Arrows = transitions
- Labels show probability (p) and reward (r)
- Self-loops = staying in same state

How do we represent MDPs mathematically?

Transition Matrix P

$P_{ij} = P(S_{t+1} = j | S_t = i)$

$\mathbf{P} =$

	S_1	S_2	S_3
S_1	0.0	1.0	0.0
S_2	0.3	0.2	0.5
S_3	0.4	0.5	0.1

Each row sums to 1.0

Reward Matrix R

$R_{ij} = E[R_{t+1} | S_t = i, S_{t+1} = j]$

$\mathbf{R} =$

	S_1	S_2	S_3
S_1	0	+5	0
S_2	−1	+4	+3
S_3	0	+1	−2

Expected reward per transition

This can be ofc. made only for discrete and finite space

How do agents choose actions?

Definition (Policy)
A policy π is a mapping from states to actions:

- **Deterministic**: $a = \pi(s)$
- **Stochastic**: $a \sim \pi(a|s)$

Example Policies:

- **Random**: Choose actions uniformly
- **Greedy**: Always go toward goal
- **Safe**: Avoid traps at all costs
- **Optimal**: Maximize cumulative reward

The goal of RL is to find the optimal policy π^*!

How do we measure total reward over time?

The Problem
How do we measure ”total reward” when actions have long-term consequences?

Simple Example: Saving Money

- Today: Save \$100 (reward = -\$100)
- Tomorrow: Earn \$5 interest (reward = +\$5)
- Next day: Earn \$5 more interest (reward = +\$5)
- ...

Question: Was saving worth it? We need to sum ALL future rewards.

Why do we discount future rewards?

Definition (Discounted Return)
The return G_t is the cumulative discounted reward from time t:

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

↳ what I expect to get in future, with respect with my "forgetting"

Why Discounting? ($\gamma < 1$)

- **Mathematical convenience**: Ensures finite returns
- **Uncertainty**: Future rewards are less certain
- **Preference**: We prefer immediate rewards
- **Computational**: Avoids infinite planning horizons

Examples: $\gamma = 0$ (myopic), $\gamma = 0.9$ (balanced), $\gamma = 1$ (far-sighted)

How do we predict future success?

Definition (State Value Function)
The value of state s under policy π is the expected return:

$$V^{\pi}(s) = \mathbb{E}[G_t | s_t = s, \pi]$$

Definition (Action Value Function (Q-function))
The value of taking action a in state s under policy π:

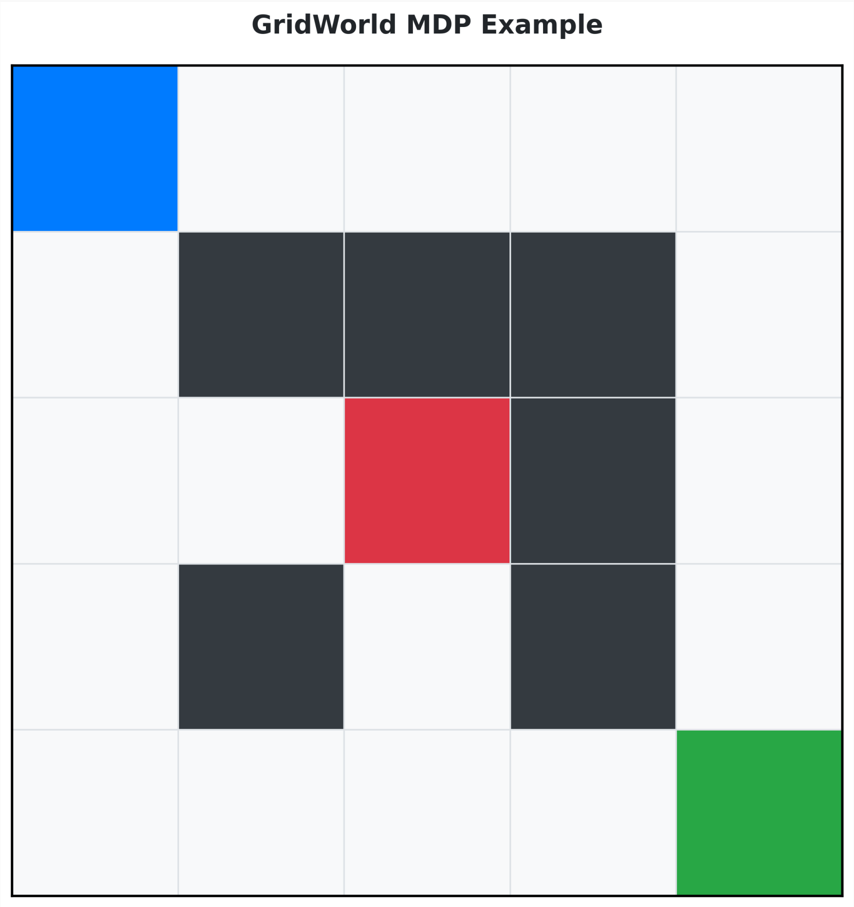
$$Q^{\pi}(s, a) = \mathbb{E}[G_t | s_t = s, a_t = a, \pi]$$

Intuition

- $V^{\pi}(s)$: "How good is it to be in state s?"
- $Q^{\pi}(s, a)$: "How good is it to take action a in state s?"
- Both measure expected future reward, not immediate reward!

What is the equivalent of Q^{π} in continuous space (like car driving)

What do value functions look like visually?

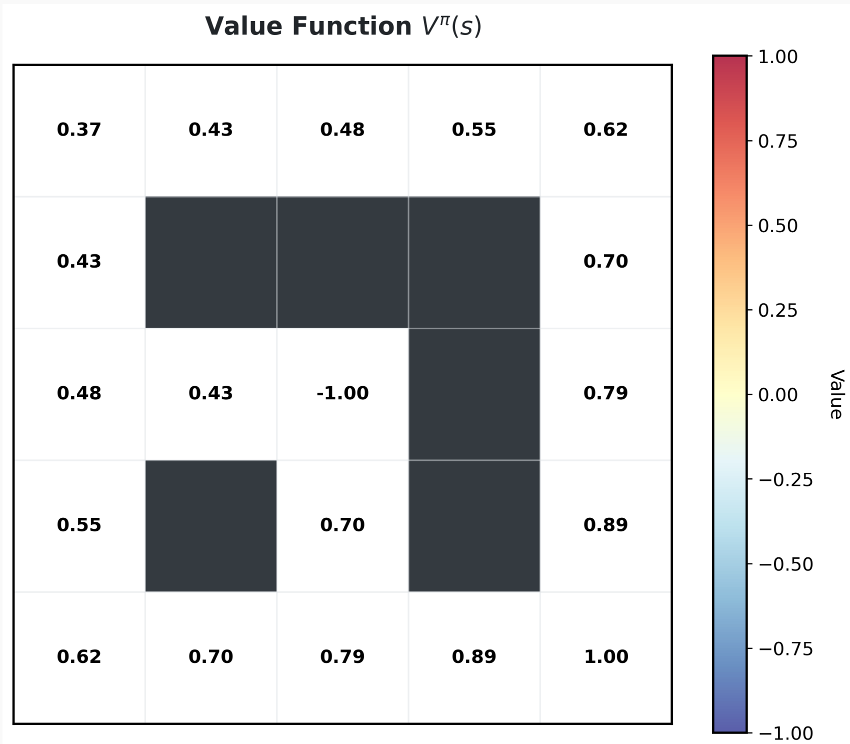


Simple 5×5 GridWorld

MDP Components:

- $\mathcal{S}$: Grid positions (i, j)
- $\mathcal{A}$: {Up, Down, Left, Right}
- P: Deterministic movement
- R: Goal=+1, Trap=-1, Step=-0.01
- $\gamma = 0.9$

How do values spread through the state space?



State values $V^\pi(s)$ for a specific policy

What We See:

- **Goal state**: High value (close to +1)
- **Trap state**: Low value (close to -1)
- **Gradient**: Values decrease with distance from goal

Key Insight: Value functions tell us which states are "good" to be in!

🎮 Mentimeter Poll

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In the GridWorld example, if $\gamma = 0$ (no discounting), what happens?

- A) The agent becomes more patient
- B) Only immediate rewards matter
- C) The Bellman equation becomes invalid
- D) All states have the same value
- E) The policy becomes deterministic

Bellman Equations

What is the fundamental relationship in value functions?

Value functions satisfy a **recursive relationship** - we can define the value of a state in terms of the values of its successor states.

Theorem (Bellman Equation)

$$
\begin{aligned}
V^\pi(s) &= \mathbb{E}[G_t | S_t = s, \pi] \\
&= \mathbb{E}[R_{t+1} + \gamma G_{t+1} | S_t = s, \pi] \\
&= \mathbb{E}[R_{t+1} + \gamma \mathbb{E}[G_{t+1} | S_{t+1} = s', \pi] | S_t, \pi] \\
&= \mathbb{E}[R_{t+1} + \gamma V^\pi(S_{t+1}) | S_t = s, \pi] \\
&= \sum_a \pi(a|s) \sum_{s',r} P(s', r | s, a)[r + \gamma V^\pi(s')]
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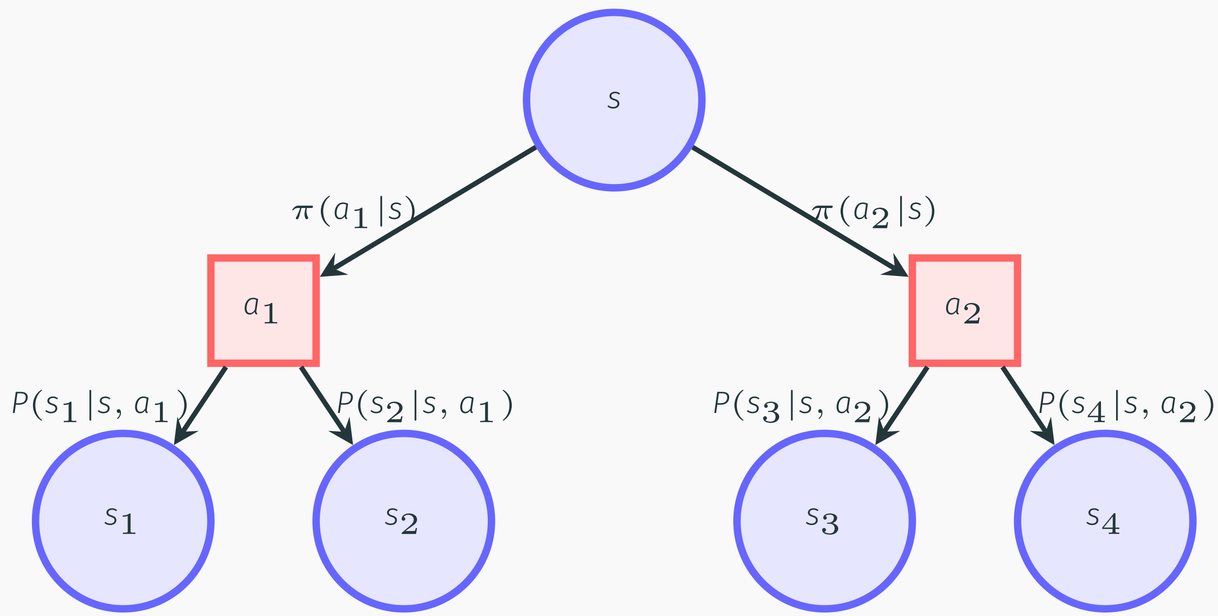
\end{aligned}
$$

This equation is the foundation of ALL reinforcement learning algorithms!

How can we visualize the Bellman equation?

Backup Diagram for $V^\pi(s)$:



Key: States (circles), Actions (squares), Values flow backwards through tree

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The Bellman equation expresses:

- A) How to compute immediate rewards
- B) The relationship between current and future values
- C) How to choose optimal actions
- D) The transition probabilities
- E) The discount factor

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Which of these is **NOT** a correct step in deriving the Bellman equation?

A) $V^{\pi}(s) = \mathbb{E}[G_t | s_t = s, \pi]$

B) $V^{\pi}(s) = \mathbb{E}[R_{t+1} + \gamma G_{t+1} | s_t = s, \pi]$

C) $V^{\pi}(s) = \sum_{a} \pi(a|s) \sum_{s'} P(s'|s,a)[r + \gamma V^{\pi}(s')]$

D) $V^{\pi}(s) = \sum_{a} \pi(a|s) \sum_{s',r} P(s', r|s,a)\gamma[r + V^{\pi}(s')]$

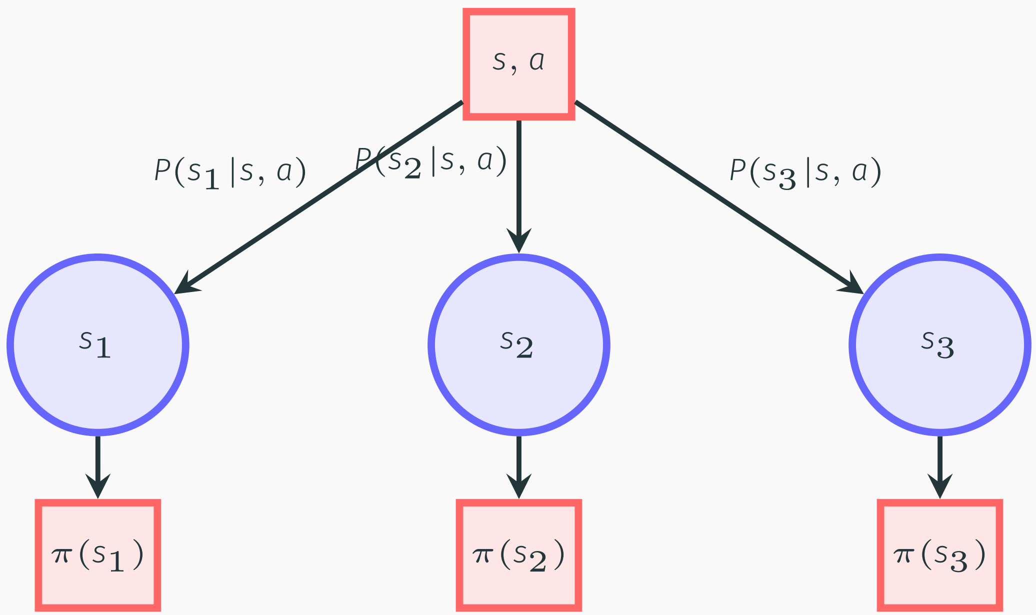
E) $V^{\pi}(s) = \sum_{a} \pi(a|s) \sum_{s',r} P(s', r|s,a)[r + \gamma V^{\pi}(s')]$

What about Bellman equations for action values?

Theorem (Bellman Equation for Action Values)
The action value function $Q^\pi(s,a)$ satisfies:

$$Q^\pi(s,a) = \sum_{s',r} P(s',r|s,a)[r + \gamma \sum_{a'} \pi(a'|s')Q^\pi(s',a')]$$

Backup Diagram for $Q^\pi(s,a)$:



What defines the best possible policy?

Definition (Optimal Value Functions)

↗ we get this by exploring the environment

$$V^*(s) = \max_\pi V^\pi(s) \quad (1)$$

$$Q^*(s,a) = \max_\pi Q^\pi(s,a) \quad (2)$$

Relationship Between Optimal Functions

$$Q^*(s,a) = \mathbb{E}[R_{t+1} + \gamma V^*(s')|s_t = s, a_t = a]$$

↗ expectation of
- what I get
- what comes in future

$$V^*(s) = \max_a Q^*(s,a)$$

Definition (Optimal Policy)

An optimal policy π^* satisfies:

$$\pi^*(s) = \text{argmax}_a Q^*(s,a)$$

How do we find the optimal value function?

We want the optimal policy π^* that maximizes value, not just evaluate a given policy.

Key insight: If we act optimally, we choose the action with highest Q-value:

$$\pi^*(s) = \text{argmax}_a Q^*(s, a)$$

this evaluates

This means: $V^*(s) = \max_a Q^*(s, a)$

this performs the action! :-)

Can we write a recursive equation for $V^*(s)$ like we did for $V^\pi(s)$?

What are the Bellman optimality equations?

The Bellman Optimality Equation expresses the optimal value function recursively:

Theorem (Bellman Optimality Equation for V^*)

$$
\begin{aligned}
V^*(s) &= \max_a Q^*(s,a) && \text{(choose best action)} \\
&= \max_a \mathbb{E}[G_t | S_t = s, A_t = a, \pi^*] && \text{(expected return under optimal policy)} \\
&= \max_a \mathbb{E}[R_{t+1} + \gamma G_{t+1} | S_t = s, A_t = a, \pi^*] && \text{(break into immediate + future)} \\
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recursion

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V^*(s) &= \max_a Q^*(s,a) && \textbf{(choose best action)} \\
&= \max_a \mathbb{E}[G_t | S_t = s, A_t = a, \pi^*] && \textbf{(expected return under optimal policy)} \\
&= \max_a \mathbb{E}[R_{t+1} + \gamma G_{t+1} | S_t = s, A_t = a, \pi^*] && \textbf{(break into immediate + future)} \\
&= \max_a \mathbb{E}[R_{t+1} + \gamma V^*(S_{t+1}) | S_t = s, A_t = a] && \textbf{(future return is optimal)} \\
&= \max_a \sum_{s',r} P(s', r | s, a)[r + \gamma V^*(s')] && \textbf{(expand expectation)}
\end{aligned}
$$

Theorem (Bellman Optimality Equation for Q^*)

$$
\begin{aligned}
Q^*(s,a) &= \mathbb{E}[R_{t+1} + \gamma \max_{a'} Q^*(s', a') | s_t = s, a_t = a] \\
&= \sum_{s',r} P(s', r | s, a)[r + \gamma \max_{a'} Q^*(s', a')]
\end{aligned}
$$

The **max operator** makes these equations nonlinear!

menti.com - Code: XXXX XXXX

If you got lost today, at what point did you lose track?

- A) Markov property and MDPs
- B) Bellman equations for state values
- C) Bellman equations for action values
- D) Backup diagrams
- E) Optimal value functions and policies
- F) Bellman optimality equations
- G) All concepts are clear

How does this connect to modern deep reinforcement learning?

The Bridge from Classical to Modern

- Same math, different representation
- Tables → Neural networks ⇝ the tables are only predicted by a network
- Exact solutions → Approximate solutions ⇝ because of predictions...
- Small problems → Complex problems ⇝ :)

Tabular RL:

- $V(s)$ stored in table
- $Q(s, a)$ stored in table
- Exact Bellman updates
- Works for small $|\mathcal{S}|, |\mathcal{A}|$

Deep RL:

- $V(s) \approx V_\theta(s)$ neural net
- $Q(s, a) \approx Q_\theta(s, a)$ neural net
- Approximate Bellman updates
- Scales to huge state spaces

What are the key concepts we've learned today?

Key Takeaways

- **MDPs**: Formalize RL problems with states, actions, rewards
- **Policies**: Define agent's behavior, can be deterministic or stochastic
- **Returns**: Measure long-term success using discounted rewards
- **Value Functions**: Predict future success from states or state-action pairs
- **Bellman Equations**: Capture recursive relationships in value functions
- **Optimal Policies**: Achieve maximum expected return, defined by optimal value functions

Next Week: Dynamic Programming - How to compute optimal policies and value functions using Bellman equations

Reading and Resources

📚 Essential Reading

- **Sutton & Barto**: Chapters 1 to 3 (MDPs and Bellman Equations)
- Focus especially on: Section 3.3 (Returns), 3.5 (Policies and Value Functions), 3.6 (Optimal Policies)