

1) $u \perp v$ Určete $\|u+v\|$ a $\|u-v\|$

$$\|u\| = 12$$

$$\|v\| = 5$$

$$\langle u, v \rangle = 0$$

$$\begin{aligned} \|u+v\| &= \sqrt{\langle u+v | u+v \rangle} = \sqrt{\langle u+v | u \rangle + \langle u+v | v \rangle} = \sqrt{\langle u | u+v \rangle + \langle v | u+v \rangle} \\ &= \sqrt{\langle u | u \rangle + \underbrace{\langle u | v \rangle}_0 + \underbrace{\langle v | u \rangle}_0 + \langle v | v \rangle} = \sqrt{\langle u | u \rangle + \langle v | v \rangle} = \\ &= \sqrt{25 + 144} = \sqrt{169} = 13 \end{aligned}$$

2) Gram-Schmidt algoritmus:

for i in n :

$$w_i = u_i - \sum_{j=1}^{i-1} \langle u_i | v_j \rangle v_j$$

$$v_i = \frac{1}{\|w_i\|} \cdot w_i$$

↗ vstup je $u_1 \dots u_n$ je báze

↘ výstup je $v_1 \dots v_n$ je ortonor. báze

$$a) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \dots \begin{matrix} u_1^T \\ u_2^T \\ u_3^T \end{matrix}$$

$$i=1: w_1 = u_1 - \sum_{j=1}^0 \dots = (1, 1, 1, 1)^T$$

$$v_1 = \frac{1}{\|w_1\|} \cdot w_1 = \frac{1}{\sqrt{4}} \cdot (1, 1, 1, 1)^T = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T$$

Finální orto
vektory

$$\langle u_2 | v_1 \rangle = \langle (1, 1, 1, 1)^T | \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T \rangle = 5$$

$$\|w_2\| = \sqrt{\frac{36}{4}} = 3$$

$$i=2: w_2 = u_2 - \sum_{j=1}^1 \langle u_2 | v_j \rangle v_j = (1, 1, 1, 1)^T - 5 \cdot \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T = \left(\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}\right)^T$$

$$v_2 = \frac{1}{\|w_2\|} \cdot w_2 = \frac{1}{3} \cdot \left(\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}\right)^T = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)^T$$

$$\langle u_3 | v_1 \rangle = \langle (1, 2, 3, 4)^T | \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T \rangle = 5$$

$$\langle u_3 | v_2 \rangle = \langle (1, 2, 3, 4)^T | \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)^T \rangle = -1$$

$$\|w_3\| = \sqrt{4} = 2$$

$$i=3: w_3 = u_3 - \langle u_3 | v_1 \rangle v_1 - \langle u_3 | v_2 \rangle v_2 = (1, 2, 3, 4)^T - 5 \cdot \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T + 1 \cdot \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)^T = (-1, -1, 1, 1)^T$$

$$v_3 = \frac{1}{\|w_3\|} \cdot w_3 = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T$$

3) Rozšířte bázi z $\mathbb{R}^3$ na $\mathbb{R}^4$, kde báze := $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}^T, \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix}^T, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}^T \right\}$

- takže chceme u_4 takovou, že $u_4 \notin R(A)$ -> tedy že je LNZ m. vektorů báze

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 0 & -3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \rightarrow \text{lze zvolit } u_4 = (0, 0, 0, 1)^T$$

$$w_4 = u_4 - \langle u_4 | v_1 \rangle v_1 - \langle u_4 | v_2 \rangle v_2 - \langle u_4 | v_3 \rangle v_3$$

$$(0, 0, 0, 1)^T - \frac{1}{2} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} - (-\frac{1}{2}) \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}^T = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}^T$$

$$v_4 = \frac{1}{2} \cdot w_4 = \frac{1}{2} \cdot w_4 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}^T$$

h) Určete ortogonální projekci p vektoru $a = (2, 2, 1, 5)^T$ do $R(A)$ a souřadnice $[p]_2$

$$P_{R(A)}(a) = \langle a | v_1 \rangle v_1 + \langle a | v_2 \rangle v_2 + \langle a | v_3 \rangle v_3 = 5 \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}^T - 2 \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix}^T + 1 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}^T =$$

Toto je ta projekce $\rightarrow (1, 1, 2, 5)^T$

$$6) \quad A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & -1 & -1 \\ 1 & -2 & 2 \end{pmatrix}$$

-> všechny jsou kolmé
- aby to byly vektory ortonormální
báze, musí mít vlastní normu rovnou 1.

$$b = (10, 5, 13, 9)^T$$

$$\|u_1\| = \sqrt{4+16+4+1} = 5$$

$$\|u_2\| = \sqrt{1+4+16+4} = 5$$

$$\|u_3\| = \sqrt{1+4} = \sqrt{5}$$

$$z_1 = \left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}, \frac{1}{5} \right)^T$$

$$z_2 = \left(\frac{1}{5}, \frac{2}{5}, -\frac{1}{5}, -\frac{2}{5} \right)^T$$

$$z_3 = \left(0, 0, \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right)^T$$

$$b' = P_2(b) = \langle b | z_1 \rangle z_1 + \langle b | z_2 \rangle z_2 + \langle b | z_3 \rangle z_3 = 15 \left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}, \frac{1}{5} \right)^T - 10 \left(\frac{1}{5}, \frac{2}{5}, -\frac{1}{5}, -\frac{2}{5} \right)^T + \sqrt{5} \left(0, 0, \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right)^T$$

$$b' = (4, 8, 13, 9)^T$$

$$x' = (15, -10, \sqrt{5})^T$$