

Jméno a příjmení: Mikoláš Fňmm, McCoolAusPotřebný čas: **MOC!**

$$\langle x, y \rangle = \sum_{i=1}^4 x_i y_i$$

1. V prostoru $\mathbb{R}^4$ se standardním skalárním součinem najděte ortonormální bázi řádkového prostoru následující matice

$$\begin{pmatrix} -4 & 1 & -2 & 2 \\ -6 & 4 & -8 & 3 \\ 8 & 3 & -6 & -4 \end{pmatrix} \sim \begin{pmatrix} -4 & 1 & -2 & 2 \\ 0 & 5 & -10 & 0 \\ -2 & 3 & -6 & 1 \end{pmatrix} \sim \begin{pmatrix} -4 & 1 & -2 & 2 \\ 0 & 5 & -10 & 0 \\ 0 & 5 & -5 & 0 \end{pmatrix} \sim \begin{pmatrix} -4 & 1 & -2 & 2 \\ 0 & 5 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \rightarrow u_1^T \\ \rightarrow u_2^T \\ \end{matrix}$$

Poté získanou bázi doplňte na ortonormální bázi celého prostoru ($\mathbb{R}^4$). $i=1:$

$$u_1 = (-4, 1, -2, 2)^T$$

$$v_1 = \frac{1}{\|u_1\|} \cdot u_1 = \frac{1}{\sqrt{16+1+4+4}} \cdot (-4, 1, -2, 2)^T = \left(-\frac{4}{5}, \frac{1}{5}, -\frac{2}{5}, \frac{2}{5}\right)^T$$

poté doplnit bázi $\mathcal{B}(A)$, např.:

$$u_3^T = (0, 0, 1, 0)^T$$

$$u_4^T = (0, 0, 0, 1)^T$$

 $i=2:$

$$u_2 = (0, 5, -10, 0)^T - \sum_{j=1}^1 \langle u_2 | v_j \rangle v_j = (0, 5, -10, 0)^T - 5 \cdot \left(-\frac{4}{5}, \frac{1}{5}, -\frac{2}{5}, \frac{2}{5}\right)^T = (4, 4, -8, -2)^T$$

$$v_2 = \frac{1}{\|u_2\|} \cdot u_2 = \frac{1}{10} \cdot (4, 4, -8, -2)^T = \left(\frac{2}{5}, \frac{2}{5}, -\frac{4}{5}, -\frac{1}{5}\right)^T$$

$$\sqrt{16+16+64+1} = \sqrt{100}$$

$$\langle u_2 | v_1 \rangle = \langle (0, 5, -10, 0) | \left(-\frac{4}{5}, \frac{1}{5}, -\frac{2}{5}, \frac{2}{5}\right) \rangle = 0 + 1 + 0 = 0$$

Ortonormální báze $\mathcal{B}(A)$ $i=3:$

$$u_3 = (0, 0, 1, 0)^T - \langle u_3 | v_1 \rangle v_1 - \langle u_3 | v_2 \rangle v_2 = (0, 0, 1, 0)^T - \left(\frac{8}{25}, -\frac{2}{25}, \frac{4}{25}, -\frac{1}{25}\right)^T - \left(-\frac{8}{25}, \frac{8}{25}, \frac{16}{25}, \frac{1}{25}\right)^T = \left(0, \frac{10}{25}, \frac{5}{25}, 0\right)^T$$

$$v_3 = \frac{1}{\|u_3\|} \cdot u_3 = \frac{1}{\sqrt{5}} \cdot \left(0, \frac{2}{5}, \frac{1}{5}, 0\right)^T = \sqrt{5} \cdot \left(0, \frac{2}{5}, \frac{1}{5}, 0\right)^T = \left(0, \frac{2\sqrt{5}}{5}, \frac{1\sqrt{5}}{5}, 0\right)^T$$

$$\sqrt{0 + \frac{4}{25} + \frac{1}{25} + 0} = \sqrt{\frac{5}{25}} = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}}$$

 $i=4:$

$$u_4 = (0, 0, 0, 1)^T - \langle u_4 | v_1 \rangle v_1 - \langle u_4 | v_2 \rangle v_2 - \langle u_4 | v_3 \rangle v_3 = (0, 0, 0, 1)^T - \left(-\frac{8}{25}, \frac{2}{25}, -\frac{4}{25}, \frac{1}{25}\right)^T - \left(-\frac{2}{25}, -\frac{2}{25}, \frac{4}{25}, \frac{1}{25}\right)^T - 0 = \left(\frac{10}{25}, 0, 0, \frac{20}{25}\right)^T$$

$$v_4 = \frac{1}{\|u_4\|} \cdot u_4 = \frac{\sqrt{5}}{2} \cdot \left(\frac{2}{5}, 0, 0, \frac{1}{5}\right)^T = \left(\frac{2\sqrt{5}}{10}, 0, 0, \frac{1\sqrt{5}}{10}\right)^T = \left(\frac{\sqrt{5}}{5}, 0, 0, \frac{2\sqrt{5}}{5}\right)^T = \left(\frac{1}{\sqrt{5}}, 0, 0, \frac{2}{\sqrt{5}}\right)^T$$

$$\sqrt{\frac{4}{25} + 0 + 0 + \frac{16}{25}} = \sqrt{\frac{20}{25}} = \frac{\sqrt{20}}{5} = \frac{2\sqrt{5}}{5} = \frac{2}{\sqrt{5}}$$

Doplnění báze do celého prostoru $\mathbb{R}^4$

2. Najděte $\mathbf{x}'$ takové, že minimalizuje $\|\mathbf{Ax}' - \mathbf{b}\|$, kde

$$\mathbf{A} = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & u_5 \\ 3 & 2 & -1 & 1 & 3 \\ 3 & 1 & 3 & -1 & -2 \\ 2 & -3 & -1 & -1 & -1 \\ 1 & -1 & -2 & -3 & 1 \\ 1 & -3 & 1 & 3 & 1 \\ 1 & 1 & -3 & 2 & -3 \end{pmatrix}, \quad \mathbf{b} = (26, 5, 34, -18, -30, -13)^T$$

→ Všechny vektory jsou opět kolmé!
Stejně tedy udělat projekci m. složek jejího normalizovaného tvaru

Určete také hodnotu $\|\mathbf{Ax}' - \mathbf{b}\|$. (Jde o normu vzhledem ke standardnímu skalárnímu součinu).

$$\|u_1\| = \sqrt{9+9+4+1+1+1} = \sqrt{25} = 5$$

$$\|u_2\| = \sqrt{4+1+9+1+9+1} = 5$$

$$v_1 = \frac{1}{5} (3, 3, 2, 1, 1, 1)^T = \left(\frac{3}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \right)^T$$

$$v_2 = \frac{1}{5} (2, 1, -3, -1, -3, 1)^T = \left(\frac{2}{5}, \frac{1}{5}, -\frac{3}{5}, -\frac{1}{5}, -\frac{3}{5}, \frac{1}{5} \right)^T$$

$$\|u_3\| = \sqrt{1+9+1+4+1+1} = 5$$

$$\|u_4\| = \sqrt{1+1+1+4+1+1} = 5$$

$$v_3 = \frac{1}{5} (-1, 2, -1, -2, 1, -3)^T = \left(-\frac{1}{5}, \frac{2}{5}, -\frac{1}{5}, -\frac{2}{5}, \frac{1}{5}, -\frac{3}{5} \right)^T$$

$$v_4 = \frac{1}{5} (1, -1, -1, -3, 2, 2)^T = \left(\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{3}{5}, \frac{2}{5}, \frac{2}{5} \right)^T$$

$$\|u_5\| = \sqrt{9+4+1+9+1+9} = 5$$

$\mathbf{x}'$ odpovídá Fourierovým koeficientům!

$$v_5 = \frac{1}{5} (3, -2, -1, 1, 1, -3)^T = \left(\frac{3}{5}, -\frac{2}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, -\frac{3}{5} \right)^T$$

$$\mathbf{x}' = (20, 10, 0, -15, 5)^T$$

$$P_2(\mathbf{b}) = \underbrace{\langle \mathbf{b} | v_1 \rangle}_{20} v_1 + \underbrace{\langle \mathbf{b} | v_2 \rangle}_{10} v_2 + \underbrace{\langle \mathbf{b} | v_3 \rangle}_{20} v_3 + \underbrace{\langle \mathbf{b} | v_4 \rangle}_{-15} v_4 + \underbrace{\langle \mathbf{b} | v_5 \rangle}_{15} v_5$$

$$= (4, 2, 0, -3, 1)^T$$

$$\mathbf{b}' = (12+4+0-3+5, 12+2+0+3-2, 8-6+0+3-1, 4-2+0+1+1, 4-6+0-1+1, 4+2+0-3-3)^T$$

$$\mathbf{b}' = (16, 15, 4, 12, -10, -3)^T$$

$$\|\mathbf{Ax}' - \mathbf{b}\| = \|(16, 15, 4, 12, -10, -3)^T - (26, 5, 34, -18, -30, -13)^T\| = \underline{\underline{50}}$$