Proof of Gaussian derivative equality:

$$\sigma \cdot \nabla^2 G \overset{?}{=} \frac{\partial G}{\partial \sigma}, \quad G(x,y,\sigma) = \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{x^2+y^2}{2\sigma^2}}$$

let $a = \frac{x^2+y^2}{2\sigma^2}$

1) $$\sigma \nabla^2 G = \sigma \cdot \left(\frac{\partial^2 G}{\partial^2 x} + \frac{\partial^2 G}{\partial^2 y}\right)$$

$(e^{-a})'$ $\quad a'$

a) $$\frac{\partial G}{\partial x} = \frac{\partial}{\partial x} \cdot \left(\frac{e^{-a}}{2\pi\sigma^2}\right) = \left(\frac{1}{2\pi\sigma^2} \cdot e^{-a}\right) \cdot \left(-\frac{1}{\cancel{2}\sigma^2} \cdot \cancel{2}x\right) = -x \cdot e^{-a} \cdot \frac{1}{2\pi\sigma^4}$$

$$\frac{\partial G}{\partial y} = \frac{\partial}{\partial y} \cdot \left(\frac{e^{-a}}{2\pi\sigma^2}\right) = \frac{1}{2\pi\sigma^2} \cdot e^{-a} \cdot -\frac{1}{2\sigma^2} \cdot 2y = -y \cdot e^{-a} \cdot \frac{1}{2\pi\sigma^4}$$

b) $$\frac{\partial^2 G}{\partial^2 x} = \frac{\partial}{\partial x} \cdot \frac{-x}{2\pi\sigma^4 \cdot e^a} = -\frac{1}{(2\pi\sigma^4 \cdot e^a)^2} \cdot \left(2\pi\sigma^4 \cdot e^a - x \cdot 2\pi\sigma^4 \cdot e^a \cdot \frac{1}{2\sigma^2} \cdot 2x\right) =$$

$$2\pi\sigma^4 \cdot a^2 \cdot \left(1 - x \cdot \frac{x}{\sigma^2}\right) = 2\pi\sigma^4 \cdot a^2 \cdot \left(1 - \frac{x^2}{\sigma^2}\right)$$

$$= -\frac{1}{2\pi\sigma^4 \cdot e^a} \cdot \left(1 - \frac{x^2}{\sigma^2}\right)$$

$$\frac{\partial^2 G}{\partial^2 y} = \ldots = -\frac{1}{2\pi\sigma^4 \cdot e^a} \cdot \left(1 - \frac{y^2}{\sigma^2}\right)$$

$$\sigma \nabla^2 G = \frac{-1}{2\pi\sigma^4 \cdot e^{\frac{x^2+y^2}{2\sigma^2}}} \cdot \left(2 - \frac{y^2}{\sigma^2} - \frac{x^2}{\sigma^2}\right)$$

2) $$\frac{\partial G}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(\frac{e^{-a}}{2\pi\sigma^2}\right) =$$

$$\frac{(x^2+y^2) \cdot \sigma^2 - 2\sigma}{2\sigma^4}$$

$$= \frac{1}{(2\pi\sigma^2)^2} \cdot \left(2\pi\sigma^2 \cdot e^{-a} \cdot \frac{(x^2+y^2) \cdot \cancel{2}\sigma^2 - (x^2+y^2) \cdot \cancelto{2}{4}\sigma}{2\cancel{4}\sigma^4} - e^{-a} \cdot 4\pi\sigma\right) =$$

$$= \frac{1}{(2\pi\sigma^2)^2} \cdot \left(2\pi\sigma^2 \cdot \left(e^{-a} \cdot \frac{(x^2+y^2) \cdot \sigma^2 - 2\sigma}{2\sigma^4} - e^{-a} \cdot \frac{2}{\sigma}\right)\right)$$

$$= \frac{1}{2\pi\sigma^2} \cdot \left(e^{-a} \cdot \frac{(x^2+y^2) \cdot \sigma^2 - 2\sigma}{2\sigma^4} - e^{-a} \cdot \frac{2}{\sigma}\right) = \frac{1}{2\pi\sigma^2 \cdot e^a} \cdot \left(\frac{(x^2+y^2) \cdot \sigma^2 - 2\sigma}{2\sigma^4} - \frac{2}{\sigma}\right)$$

3) Left for reader...