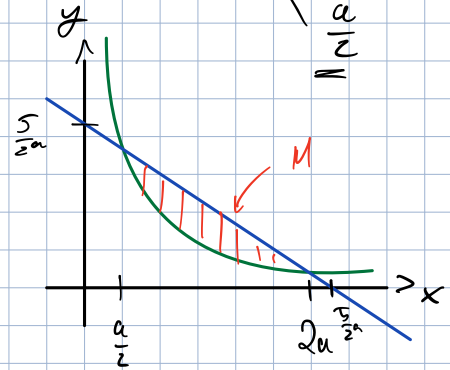
Pokud $M \subseteq \mathbb{R}^n$, tak $\int_M 1$ je „n-rozměrný objem" $M$ (pokud integrál existuje)

**Př:** Vypočítejte obsah rovinné plochy ohraničené křivkami $xy=a^2$, $x+y=\frac{5}{2}a$, $a>0$:

$- \; y=\frac{a^2}{x}$ — můžu vydělit, protože $a^2 = xy > 0$.

1) Najdu průsečíky, které mi budou vymezovat oblast

$$xy=a^2, \quad x+y=\tfrac{5}{2}a \;\to\; y=\tfrac{5}{2}a-x \;\to\; \tfrac{5}{2}ax-x^2=a^2 \;\to\; x^2-\tfrac{5}{2}ax+a^2=0 \;\to\; \frac{\frac{5}{2}a \pm \sqrt{\frac{25}{4}a^2-4a^2}}{2} \;\to\; x_{1,2} = \begin{cases} 2a \\ \frac{a}{2} \end{cases}$$




2) Spočítám obsah

$$S(M) \overset{\text{Fubini}}{=} \int_{\frac{a}{2}}^{2a} \int_{\frac{a^2}{x}}^{\frac{5}{2}a-x} 1 \, dy\, dx = \int_{\frac{a}{2}}^{2a} [y]_{\frac{a^2}{x}}^{\frac{5}{2}a-x} dx = \int_{\frac{a}{2}}^{2a} \frac{5}{2}a - x - \frac{a^2}{x}\, dx = \left[\frac{5}{2}ax - \frac{x^2}{2} - a^2 \cdot \log|x|\right]_{\frac{a}{2}}^{2a}$$

$$= \left(5a^2 - 2a^2 - a^2 \cdot \log(2a)\right) - \left(\frac{5}{4}a^2 - \frac{a^2}{8} - a^2 \cdot \log\left(\frac{a}{2}\right)\right) = \left(3a^2 - a^2 \cdot \log(2a)\right) - \left(\frac{9}{8}a^2 - a^2 \cdot \log\left(\frac{a}{2}\right)\right) =$$

$$= 3a^2 - a^2 \cdot \log(2a) - \frac{9}{8}a^2 + a^2 \cdot \log\left(\frac{a}{2}\right) = \frac{15}{8}a^2 - a^2 \cdot \left(\log(2a) - \log\left(\frac{a}{2}\right)\right) = \frac{15}{8}a^2 - a^2 \cdot \log\left(\frac{2a}{\frac{a}{2}}\right)$$

$$= \underline{\underline{a^2 \cdot \left(\frac{15}{8} - \log(4)\right)}}.$$

$y = \pm\sqrt{px+p^2}$

**Př:** Nalezněte obsah plochy dané křivkami $y^2=2px+p^2$, $y^2=-2qx+q^2$, $p,q>0$

1) Naleznu průsečíky:

$y^2=2px+p^2$

$y^2=-2qx+q^2$

$2px+p^2 = -2qx+q^2$

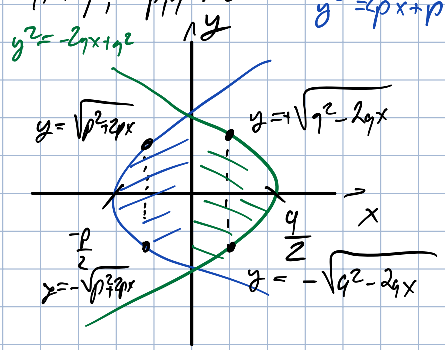
$2x\cdot(p+q)+p^2-q^2=0$

$2x\cdot(p+q)+(p-q)\cdot(p+q)=0$

$\underset{>0}{(p+q)}\cdot(2x+p-q)=0 \iff 2x+p-q=0 \Rightarrow \boxed{x=\frac{q-p}{2}}$

$0=2px+p^2 \Rightarrow x=-\frac{p}{2}$

$0=-2qx+q^2 \Rightarrow x=\frac{q}{2}$




→ je tedy sudá

→ Musíme rozdělit na dvě poloviny $M_1, M_2$

$$\int \sqrt{u} = \frac{2}{3} u^{\frac{3}{2}}\, du = \frac{2}{3}\frac{(p^2+2px)(2p\,dx)}{2p}$$

substituce $du = 2p\,dx$, $u = p^2+2px$

$$S(M) = S(M_1) + S(M_2)$$

$$S(M_1) = \int_{-\frac{p}{2}}^{\frac{q-p}{2}} \int_{-\sqrt{p^2+2px}}^{+\sqrt{p^2+2px}} 1\, dx\,dy = \int_{-\frac{p}{2}}^{\frac{q-p}{2}} \Big[y\Big]_{-\sqrt{p^2+2px}}^{\sqrt{p^2+2px}} dx = \int_{-\frac{p}{2}}^{\frac{q-p}{2}} 2\sqrt{p^2+2px}\, dx = 2\cdot\left[\frac{2}{3}\cdot\frac{(p^2+2px)^{\frac{3}{2}}}{2p}\right]_{-\frac{p}{2}}^{\frac{q-p}{2}} =$$

$p^2-p^2=0$

$$= \cancel{2}\cdot\frac{2}{3}\cdot\left(\left(\frac{\left(p^2+\cancel{2}p\cdot\left(\frac{q-p}{\cancel{2}}\right)\right)^{\frac{2}{3}}}{\cancel{2}p}\right) - \cancel{\left(\frac{\left(p^2+2p\cdot\left(-\frac{p}{2}\right)\right)^{\frac{3}{2}}}{\cancel{2}p}\right)}\right) = \frac{3}{2}\frac{(pq)^{\frac{3}{2}}}{p}$$

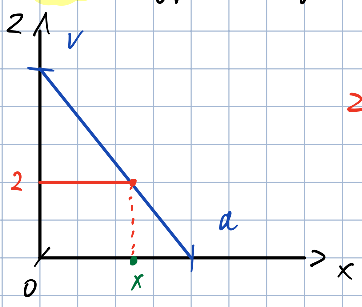
$$S(M_2) = \int_{\frac{q-p}{2}}^{\frac{q}{2}} \int_{-\sqrt{q^2-2qx}}^{\sqrt{q^2-2qx}} 1\, dy\,dx = \int_{\frac{q-p}{2}}^{\frac{q}{2}} \Big[y\Big]_{-\sqrt{q^2-2qx}}^{\sqrt{q^2-2qx}} dx = 2\cdot\int_{\frac{q-p}{2}}^{\frac{q}{2}} \sqrt{q^2-2qx} = \cancel{2}\cdot\left[\frac{2}{3}\,\frac{(q^2-2qx)^{\frac{3}{2}}}{-\cancel{2}q}\right]_{\frac{q-p}{2}}^{\frac{q}{2}} =$$

$q^2-q^2=0$

$$= \frac{2}{3}\cdot\left(\cancel{\left(\frac{\left(q^2-\cancel{2}q\cdot\frac{q}{\cancel{2}}\right)^{\frac{3}{2}}}{-q}\right)} - \left(\frac{\left(q^2-\cancel{2}q\cdot\left(\frac{q-p}{\cancel{2}}\right)\right)^{\frac{3}{2}}}{-q}\right)\right) = \frac{3}{2}\cdot\frac{(pq)^{\frac{3}{2}}}{q}$$

$$S(M) = \frac{3}{2}\cdot\left(\frac{(pq)^{\frac{3}{2}}}{q} + \frac{(pq)^{\frac{3}{2}}}{p}\right) = \frac{3}{2}\cdot(pq)^{\frac{3}{2}}\cdot\left(\frac{1}{q}+\frac{1}{p}\right) = \frac{2}{3}(pq)^{\frac{3}{2}}\cdot\left(\frac{p+q}{p\cdot q}\right) = \boxed{\frac{2}{3}\sqrt{p\cdot q}\cdot(p+q)}$$

**Př.:** Vypočtěte objem jehlanu s vrcholy v bodech $[0,0,v]$, $[0,\frac{a}{2},0]$, $[0,-\frac{a}{2},0]$, $[a,\frac{a}{2},0]$, $[a,-\frac{a}{2},0]$

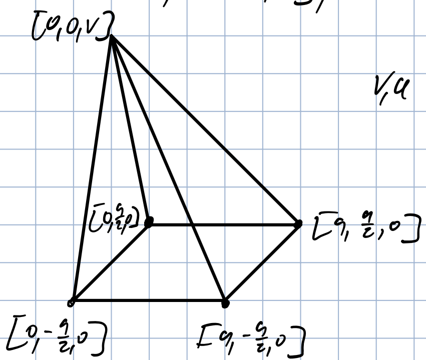



$z = \frac{v}{a}\cdot x$

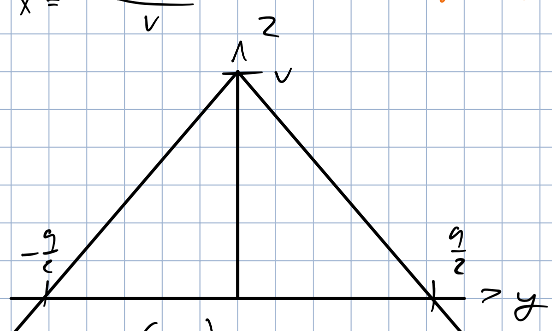
$z\cdot\frac{a}{v} = x$

potřebuji ale $(v-z)$, rozmyslet!

$x = \frac{(v-z)\cdot a}{v}$




$v, a > 0$




$y = \frac{(v-z)\cdot a}{2v}$

$$V(J) = \int_J 1 = \int_0^v \int_{-\frac{a\cdot(v-z)}{2v}}^{\frac{a(v-z)}{2v}} \int_0^{\frac{a(v-z)}{v}} 1 \, dxdydz = \int_0^v \int_{-\frac{a(v-z)}{2v}}^{\frac{a(v-z)}{2v}} \frac{a\cdot(v-z)}{v} \, dydz =$$

$$= \int_0^v \frac{a\cdot(v-z)}{v} \cdot \frac{a\cdot(v-z)}{v} \, dz = \int_0^v \overbrace{\left(\frac{a\cdot(v-z)}{v}\right)^2}^{\text{Tohle hezky ukazuje, že podstava je čtverec}} dz = \left(\frac{a}{v}\right)^2 \cdot \left[-\frac{(v-z)^3}{3}\right]_0^v = \left(\frac{a}{v}\right)^2 \cdot \left(-0 + \frac{v^3}{3}\right)$$

$$= \underline{\underline{\frac{a^2 v}{3}}}$$