Normal-form game: $(P, A, u) :=$

P = finite set of players

A = set of action profiles

$A_i \times \dots \times A_n$, where A_i := action set of player i

u - utility function

$(u_1 \dots u_n)$ where $u_i : A \to \mathbb{R}$ is utility function of i

$\forall$ player i: knows P, A, u

selects action from A

does not know actions taken by other players

receives utility $u_i(a)$ for the result of $a \in A$

- Example: kámen-nůžky-papír

Chess ?

two players

actions -> all possible situations in chess

- crazily huge

Strategy:

Mixed strategy - prob. distr. s_i on A_i

= assigns $s_i(a_i) \geq 0$ to each $a_i \in A_i$, $\sum_{a_i \in A_i} s_i(a_i) = 1$

$S_i = \{$ all mixed strategies of $i \}$

Pure strategy = special case of mixed strategy: 1 action with prob = 1

$s = (s_1 \_ s_n)$

Let $s \in S$ is mixed strategy profile.

$S_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots s_n)$

$$u_i(s) = \sum_{a \in A} u_i(a) \cdot \prod_{j=1}^{n} s_j(a_j)$$

$u_i(s_i', s_i) = S_{-i} \cdot S_i = (s_1, \dots, s_{i-1}, s_i', s_{i+1}, \dots s_n)$

$u(s_i, s_{-i}) = u(s)$

Linearity of $u_i(s)$

$$u_i(s) = \sum s_i(a_i) \cdot u_i(a_i, s_{-i})$$

Example: matching pannies

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

head/tail

NE: $\left(\frac{1}{2}T + \frac{1}{2}H, \frac{1}{2}T + \frac{1}{2}H\right)$

→ zero-sum game

↳ whatever one gets, the other loses

$\forall a \in A: u_1(a) = -u_2(a)$

: prisons dilema

	T	S
T	(-2, -2)	(0, -3)
S	(-3, 0)	(-1, -1)

T: testify
S: silent

NE: (T, T)

: battle of sexes

H: husband   W: wife
F: go to football, O: go to opera

H \ W	F	O
F	(2, 1)	(0, 0)
O	(0, 0)	(1, 2)

NE: $(F,F), (O,O), \left(\frac{2}{3}F + \frac{1}{3}O, \frac{1}{3}F + \frac{2}{3}O\right)$

: game of chicken

H \ W	T	S
T	(0, 0)	(-1, 1)
S	(1, -1)	(-10, -10)

T: turn, S: straight

- who turns, is the looser
- this is NOT zerosum game

Solution concepts:

- $i \in P$, i sees s_{-i} (the strategy of other players)

↳ then plays BEST RESPONSE of i to s_{-i} = mixed strategy s_i^* s.t. $\forall s_i' \in S_i : u_i(s_i^*, s_{-i}) \geq u_i(s_i', s_{-i})$

Nash equilibrium = mixed strategy profile $s \in S$ s.t. $\forall i \in P$: s_i is best response of i to s_{-i}

Nash Theorem

$\forall$ normal-form game has NE.

Dk:
only preparation for next lecture

Compact set:

$X \subset \mathbb{R}^d$ is compact, if it is bounded and close.

Convex set:

$\forall x, y \in X$, line segment $xy \in X$

Simplex $\Delta = \Delta(x_i, x_n) = \{ \sum_{i=1}^{n} t_i x_i : t_i \in [0,1], \sum_i^n t_i = 1 \}$

Lemma

$U_1 \dots U_n$ compact sets, $U_i \subseteq \mathbb{R}^{d_i}$, then $U = U_1 \times U_2 \dots \times U_n \subseteq \mathbb{R}^{d_1 + \dots + d_n}$ is compact.

Brouwers point thm:

$d \in \mathbb{N}$, U is non-empty compact convex set in $\mathbb{R}^d$ and $f : U \to U$ continuos function, then $\exists x_i \in U : f(x_i) = x_i$

Brouwer => Nash:

Game $G_1 = (P, A, u)$, want = NE

this is compact and convex

$S_i = \{ \text{prob. dist. } A_i \} = \Delta(a_i \in A_i)$
↳ interpret as points in $\mathbb{R}^{|A_i|}$

$S = S_1 \times \dots \times S_n$ = cartesian product of simplices

by Lemma: S is also compact.

Consider $s \in S$, we want to define $f(s) = s$ via $f : S \to S$ s.t. f is continuos and s is NE <=> $f(s) = s$, then Brouwer -> NE

Lets define f.

Take $s \in S$, we want $f(s)$.

- $\forall i \in P\ \forall a_i \in A_i$, we define $\varphi_{i,a_i} : S \to \mathbb{R}$
  by setting $\varphi_{i,a_i}(s) = \max\{0, u_i(a_i; s_{-i}) - u_i(s)\}$

- we set $$s_i'(a_i) = \frac{s_i(a_i) + \varphi_{i,a_i}(s)}{\sum_{b_i \in A_i} (s_i(b_i) + \varphi_{i,b_i}(s))} = \frac{s_i(a_i) + \varphi_{i,a_i}(s)}{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}$$

- then $s_i' \in S_i$ and $s' = (s_1', \dots, s_n') \in S$
- we set $f(s) = s'$ → f is continuos

- remains to verify that fixed points of f are NE.

because s_i is best response to s_{-i}

1) NE is fixed point of f

Let S be NE. Consider $i \in P$. S is NE $\Rightarrow \forall a_i \in A_i : \varphi_{i,a_i}(s) = 0$

Therefore $s_i'(a_i) = s_i(a_i) \Rightarrow s = s' = f(s) \Rightarrow$ <u>s is fixed point.</u>

$$s_i'(a_i) = \frac{s_i(a_i) + \underbrace{\varphi_{i,a_i}(s)}_{=0}}{\underbrace{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}_{=1}}$$

1) Fixed point of f are NE:

Let $s \in S$ such that $s = f(s)$. Take $i \in P$.

$\exists a_i' \in A$ such that $s_i(a_i') > 0$ and $u_i(a_i', s_{-i}) \leq u_i(s)$

- follow from linearity of $u_i(s)$ as otherwise

$$u_i(s) < \sum_{b_i \in A_i} u_i(b_i; s_{-i}) \cdot s_i(b_i)$$

← but here should be „=“

$$u_i(a_i', s_{-i}) \leq u_i(s) \Rightarrow \varphi_{i,a_i'}(s) = 0 \Rightarrow s_i(a_i') = s_i'(a_i') = \frac{s_i(a_i') + \boxed{\varphi_{i,a_i'}(s)}_{\to 0}}{\underbrace{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}_{=1}}$$

$\Rightarrow \forall b_i \in A_i : \varphi_{i,b_i}(s) = 0$

This suffices to show that s is NE. (There is no any improvement)

- because $\forall_i\ s''_i \in S_i:\ u_i(s''_i, s_{-i}) \underset{\text{lienarity}}{=} \sum_{b_i \in A_i} \underbrace{u_i(b_i, s_{-i})}_{\le u_i(s_i, s_{-i})} \cdot s_i(b_i) \le \underbrace{\left(\sum_{b_i \in A_i} s_i(b_i)\right)}_{=1} \cdot u_i(s) = u_i(s)$

$\boxed{\times}$

Pareto optimality

Let $G = (P, A, u)$ be a game. $s, s' \in S$.

s pareto dominates s' if:

$\forall i \in P:\ u_i(s) \ge u_i(s')$ AND

$\exists j \in P:\ u_j(s) > u_j(s')$.

s is pareto optimal if $\nexists$ s' that pareto dominates.

$\exists$ NE that is not pareto optimal. (thats why it is a paradox)

$\exists$ pareto optimal states that are not NE.

Pareto principle: Roughly 80% of outcomes consequences from 20% causes.

NE in zeroSum games

$P = \{1, 2\}$, $A_1 = \{a_1, \dots, a_m\}$ → a_i – ith possible action, not action of player i.

$A_2 = \{b_1, \dots, b_n\}$ → action for each player

Payoff matrix $(M)_{ij} = u_1(a_i, b_j) = -u_2(a_i, b_j)$

$s_1(a_i) = x_i$ => mixed strategy vectors $x = (x_1, \dots, x_m)$

$s_2(b_j) = y_j$ $\qquad y = (y_1, \dots, y_n)$

$- u_1(s_1, s_2) = \sum_{(a_i, b_j) \in A_1 \times A_2} \underbrace{u_1(a_i, b_j)}_{M_{ij}} \cdot \underbrace{s(a_i)}_{x_i} \cdot \underbrace{s(b_j)}_{y_j} = \sum_i^m \sum_j^n x_i M_{ij} y_j = \underline{\underline{x^T M y}}$

Player 1 wants to maxime payoff : $x^T M y$

Player 2 wants to minimize costs : $x^T M y$.

Consider fixed x: Best payoff 2 to x is $\min_{y \in S_2} (x^T M y) = \beta(x)$

Consider fixed y: Best payoff 1 to y is $\max_{x \in S_1} (x^T M y) = \alpha(y)$

Player 1 plays against perfect opponent $\Rightarrow$ player 1 wants to play $\bar{x}$ that satisfies $\beta(\bar{x}) = \max_{x \in S_1} \beta(x)$

$\hookrightarrow$ Worst-case optimal strategy (WOS)

Player 2 plays against perfect opponent $\Rightarrow$ wants to play $\bar{y}$ that satisfies $\alpha(\bar{y}) = \min_{y \in S_2} \alpha(y)$

(x,y) is NE if $\beta(x) = x^T M y = \alpha(y)$

Lemma:

a) $\forall x \in S_1\ \forall y \in S_2$: $\beta(x) \leq x^T M y \leq \alpha(y)$

b) If (x^*, y^*) is NE, then x^* and y^* are WOS.

c) If $\beta(x^*) = \alpha(y^*) \Rightarrow (x^*, y^*)$ is NE.

Dk:

a) $\beta(x) = \min_{y' \in S_2} x^T M y' \leq x^T M y \leq \max_{x' \in S_1} x'^T M y = \alpha(y)$

b) (x^*, y^*) is NE, a) $\Rightarrow \forall x \in S_1: \beta(x) \leq \alpha(y^*) = \beta(x^*) \Rightarrow x^*$ is WOS.

$\hookrightarrow \Rightarrow \beta(x^*) = \alpha(y^*)$

- similar to y^*.

c) a) $\Rightarrow \beta(x^*) \leq (x^*)^T M y^* \leq \alpha(y^*) \Rightarrow \beta(x^*) = (x^*)^T M y^* = \alpha(y^*)$,

therefore (x^*, y^*) is NE.

Minimax theorem:

$\forall$ zero-sum $G = (P, A, u)$, WOS exists and can be found efficiently using LP (linear programming).

$\exists v \in \mathbb{R}, \forall (x^*, y^*)$ of WOS: (x^*, y^*) is NE and $\beta(x^*) = v = \alpha(y^*)$

↳ Basically it says that when playing zero-sum game, we can reveal our strategy and it changes nothing.

How NOT to prove minimax using LP

$\max \beta(x_1 \dots x_m) = \max_x \min_y x^T M y$ ↗ this is NOT linear

$x_1 + \dots + x_n = 1, \quad x_1 \dots x_n \geq 0$ (I have prob. distr. now)

Tudy ne! :)

Linear Programming:

$$\max c^T x \qquad c, x \in \mathbb{R}^m$$

$$(P) \quad Ax \leq b \qquad b \in \mathbb{R}^n$$

$$x \geq 0 \qquad A \in \mathbb{R}^{n \times m}$$

LP

↓

$$\min b^T y \qquad y \in \mathbb{R}^n$$

$$(D) \quad A^T y \geq c$$

$$y \geq 0$$

Duality theorem:

If P, D have feasible solutions, then $\exists$ optimum of P, D (x^*, y^*) satisfying $c^T x^* = b^T y^*$