Single item auction

- 1 seller, n biders, selling single item
- ∀ bidder i has private valuation v_i
  - v_i = „how much i truly values item"
- ∀ bidder i privately gives bid b_i to the seller
- seller decides the winner (if any)
- seller selects payment p
- ∀ bidder i has utility $\begin{cases} 0 & \text{if loses} \\ v_i - p & \text{if wins} \end{cases}$

first price option has same utility as losing.

- social surplus $= \sum_{i=1}^{n} x_i \cdot v_i$ where $x_i = \begin{cases} 0 & \text{if loses} \\ 1 & \text{if wins} \end{cases}$

Awsome action properties

1) DSIC property:

∀ bidder has a dominant strategy (strategy that maximizes i's utility no matter what others do) to bid truthfully (bid $b_i = v_i$) and if ∀i bids truthfully, then $u_i \geq 0$

2) Strong performance:

If bidders bid truthfully, then social surplus is maximized.

3) Computational efficiency:

Actions runs in polynomial time.

Auction examples

1) Vickrey's auction

- winner is the highest bidder i
- payment is $p = \max_{j \neq i} b_j$ (second highest bid)

Thm:

This auction is awsome.

Dk:

1) Fix i, set $B = \max_{j \neq i} b_j$

if $b_i < B$, then i loses and $u_i = MAX(0, V_i - B) = 0$

if $b_i \geq B$, then i wins and $u_i = MAX(V_i - B, 0) = V_i - B$

if $v_i < B$, then $u_i = MAX(V_i - B, 0) = 0$

when i bids truthfully

if $v_i \geq B$, then $u_i = MAX(V_i - B, 0) = V_i - B$

also when i bids truthfully

Therefore bidding truthfully is a dominant strategy

$\rightarrow u_i \geq 0 \rightarrow$ DISC ✓

2) If bidders bid truthfully, the social surplus is equal to v_i of the highest bidder, which holds $v_i = b_i \Rightarrow$ soc. surplus is maximized ✓

3) Runs in linear time. ✓

☒

Single parameter environment

- 1 seller, n bidders, set $X \subseteq \mathbb{R}^d$ of feasible outcomes

  $x = (x_1, x_2, \dots, x_n) \in X$ where x_i = amount of goods i is interested in.
- ∀ bidder i has a private valuation v_i = how much i values a unit of good
- 1) ∀ bidder privately tells his bid b_i to the seller
- 2) Based on the bids $b = (b_1, \dots, b_n)$ seller selects the allocation $x(b) = (x_1(b), \dots, x_n(b)) \in X$
- 3) Based on the bids $b = (b_1, \dots, b_n)$ seller sets payment $p(b) = (p_1(b), \dots, p_n(b)) \in \mathbb{R}^n$
  - we assume $\forall i: p_i(b) \in \langle 0, b_i \cdot x_i(b) \rangle$

utility $u_i = v_i \cdot x_i(b) - p_i(b)$

how much he pays

the amount he gets

how much he values it

- (x, p) – mechanism

Example 1: single item auction

$$X = \left\{ x \in \{0,1\}^n : \sum_{i=1}^{n} x_i \leq 1 \right\}$$

- this models single item auction, at most one bidder gets one unit of good.

Example 2: sponsored search auction → google search example

$\alpha_1 > \dots > \alpha_n > 0$ → click-through rates

$$X = \left\{ x \in \{0, \alpha_1, \dots, \alpha_n\}^n : \underbrace{x_i = x_j \Rightarrow x_i = 0 = x_j}_{\text{every } \alpha_i \text{ used just once}} \right\}$$

- the value of slot j to i is: $v_i \cdot \alpha_j$

Allocation rule x is implementable if ∃ payment rule p such that (x, p) is DSIC.

x is monotone if $\forall i \; \forall b_{-i}$ function $x_i(b_i, b_{-i})$ is non-decreasing

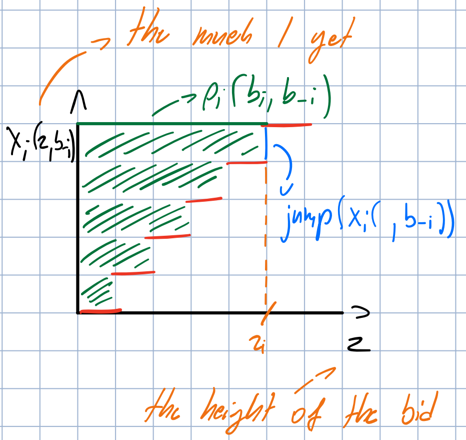
– „the more you bid, the more you win"

Myerson thm: ∀ single-parameter env. satisfies:

1) x is monotone <=> x is implementable
2) ∀ monotone x ∃! payment p s.t. (x,p) is DSIC

   ↳ true only if $(b_i = 0 \Rightarrow p_i(b) = 0)$

3) $p_i(b_i, b_{-i}) = \int_0^{b_i} z \cdot \frac{\partial}{\partial z} x_i(z, b_{-i})\, dz$



For us, typically, $x_i(z, b_{-i})$ will be piecewise constant:

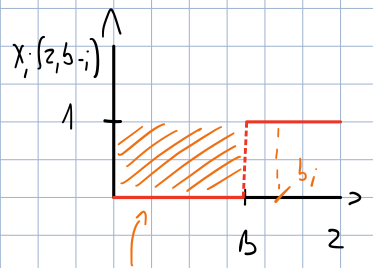
– then the payment formula:

$$p_i(b_i, b_{-i}) = \sum_{j=1}^{l} z_j \cdot \text{jump}(x_i(\,, b_{-i}))$$

where $z_1 - z_n$ are breaking points on $\langle 0, b_i \rangle$

Application: Single item auction

fix i, $B = \underset{i \neq j}{\text{MAX}}\, b_j$



$p_i(b_i, b_{-i}) = 1 \cdot B$

Application: Sponsored search

- i-th best slot gives to i-th highest bidder
- $b_1 > b_2 > b_3 > \dots > b_n$

$$\Rightarrow p_i \sum_{j=i}^{h} b_{j+1} (\alpha_j - \alpha_{j+1})$$

h=2