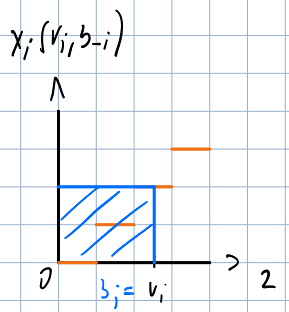
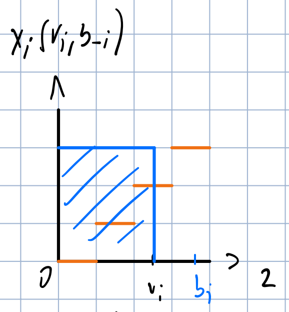
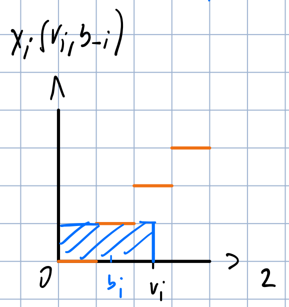
Myerson's lemma







Sketch of the proof of Myerson's lemma (Theorem 3.8). We prove all three claims at once. First, let x be an allocation rule and p be a payment rule. We recall that the mechanism (x, p) is DSIC if every bidder maximizes his utility by setting $b_i = v_i$ and, moreover, his utility $u_i(b) = v_i \cdot x_i(b) - p_i(b)$ is guaranteed to be non-negative. Expressing the utility of bidder i as a function of his bid z, we write $u_i(z; b_{-i}) = v_i \cdot x_i(z; b_{-i}) - p_i(z; b_{-i})$.

Assume the mechanism (x, p) is DSIC. The DSIC property says $u_i(z; b_{-i}) = v_i \cdot x_i(z; b_{-i}) - p_i(z; b_{-i}) \leq v_i \cdot x_i(v_i; b_{-i}) - p_i(v_i; b_{-i})$ for every z. We use a clever swapping trick. For two possible bids y and z with $0 \leq y < z$, bidder i might as well have private valuation z and can submit the false bid y if he wants, thus the DSIC condition gives

$$u_i(y; b_{-i}) = z \cdot x_i(y; b_{-i}) - p_i(y; b_{-i}) \leq z \cdot x_i(z; b_{-i}) - p_i(z; b_{-i}) = u_i(z; b_{-i}). \tag{3.1}$$

Analogously, bidder i may have his private valuation $v_i = y$ and can submit the false bid z and thus the mechanism (x, p) must satisfy

$$u_i(z; b_{-i}) = y \cdot x_i(z; b_{-i}) - p_i(z; b_{-i}) \leq y \cdot x_i(y; b_{-i}) - p_i(y; b_{-i}) = u_i(y; b_{-i}). \tag{3.2}$$

By rearranging inequalities (3.1) and (3.2) and putting them together, we obtain the following inequality called the *payment difference sandwich*:

$$z(x_i(y; b_{-i}) - x_i(z; b_{-i})) \leq p_i(y; b_{-i}) - p_i(z; b_{-i}) \leq y(x_i(y; b_{-i}) - x_i(z; b_{-i})). \tag{3.3}$$

Since $0 \leq y < z$, we obtain, by ignoring the middle part of this inequality, $x_i(y; b_{-i}) \leq x_i(z; b_{-i})$. Thus, if the mechanism (x, p) is DSIC, then x is monotone.

In the rest of the proof, we assume that the mechanism x is monotone. Let i and b_{-i} be fixed, so, in particular, we consider x_i and p_i as functions of z. First, we also assume that the function x_i is piecewise constant. Thus, the graph of x_i consists of a finite number of intervals with "jumps" between consecutive intervals; see Figure 3.1.

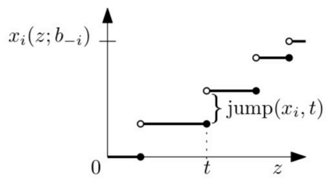


Figure 3.1: A piecewise constant function.

For a piecewise constant function f, we use $\mathrm{jump}(f, t)$ to denote the magnitude of the jump of f at point t. If we fix z in the "payment difference sandwich" inequality (3.3) and let y approach z from below in this inequality, then both sides become 0 if there is no jump of x_i at z (that is, if $\mathrm{jump}(x_i, z) = 0$). If $\mathrm{jump}(x_i, z) = h > 0$, then both sides tend to $z \cdot h$. Thus, if the mechanism (x, p) is supposed to be DSIC, then the following constraint on p must hold for every z:

$$\mathrm{jump}(p_i, z) = z \cdot \mathrm{jump}(x_i, z).$$

If we combine this constraint with the initial condition $p_i(0; b_{-i}) = 0$, we obtain a formula for the payment function p for every bidder i and bids b_{-i} of other bidders,

$$p_i(b_i; b_{-i}) = \sum_{j=1}^{\ell} z_j \cdot \mathrm{jump}(x_i(\,\cdot\,; b_{-i}), z_j), \tag{3.4}$$

where $z_1, \ldots, z_\ell$ are the breakpoints of the allocation function $x_i(\,\cdot\,; b_{-i})$ in the interval $[0, b_i]$.

With some additional facts from calculus, this argument can be generalized to general monotone functions x_i. We omit the details here and only sketch the idea for differentiable x_i.

If we divide the "payment difference sandwich" inequality (3.3) by $z - y$ and take the limit of the resulting function as z approaches y from above, then we obtain the constraint

$$p_i'(y; b_{-i}) = y \cdot x_i'(y; b_{-i}).$$

Combining this constraint with the initial condition $p_i(0; b_{-i}) = 0$, we obtain the formula

$$p_i(b_i; b_{-i}) = \int_0^{b_i} z \cdot \frac{\mathrm{d}}{\mathrm{d}z} x_i(z; b_{-i}) \, dz$$

for every z. Note that we showed that this is the only possibility for the function p if we want to extend the allocation rule x to a DSIC mechanism (x, p).

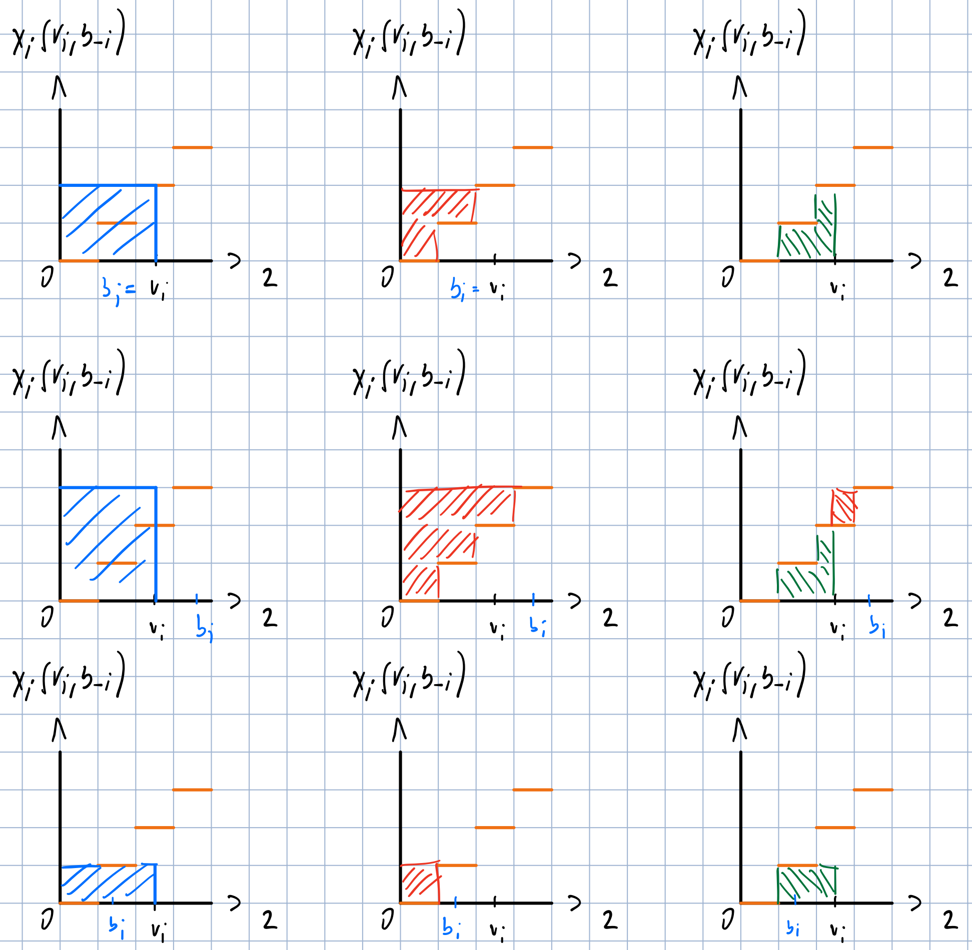
It remains to show that if x is monotone, then the mechanism (x, p) is indeed DSIC. This argument works also for monotone allocation rules that are not necessary piecewise constant. However, for the sake of clarity, we present it only for piecewise constant functions. The proof is illustrated in Figure 3.2. We recall that the utility of bidder i satisfies $u_i(b_i; b_{-i}) = v_i \cdot x_i(b_i; b_{-i}) - p_i(b_i; b_{-i})$ when he bids b_i and the other bidders bid b_{-i}. The value $v_i \cdot x_i(b_i; b_{-i})$ is represented by a blue rectangle in Figure 3.2. Using the expression (3.4), we see that the payment $p_i(b_i; b_{-i})$ of bidder i corresponds to the part of $[0, b_i] \times [0, x_i(b_i; b_{-i})]$ lying to the left of the curve $x_i(\,\cdot\,; b_{-i})$; this is represented by the red areas in Figure 3.2. Clearly, it is optimal for bidder i to bid $b_i = v_i$. Otherwise he either overbids $b_i > v_i$, in which case his utility is smaller by the area above $x_i(\,\cdot\,; b_{-i})$ in the range $[v_i, b_i]$, or he underbids $b_i < v_i$, in which case his utility is smaller by the area below $x_i(\,\cdot\,; b_{-i})$ in the range $[0, v_i]$; see Figure 3.2. □

$$u_i(z_{-i}, b_{-i}) = v_i \cdot x_i(z_i, b_{-i}) - p_i(z_i, b_i)$$

truthful bidding

overbidding

underbidding



this is my utility

Knapsack auctions

$\forall$ bidder has valuation v_i and size w_i, seller has capacity W

$$X = \{ x \in \{0,1\}^n : \sum_{i=1}^{n} x_i w_i \le W \}$$

- we want design (x, p) such that we have
  - a) DSIC
  - b) Max. Soc. Surplus $\sum_{i=1}^{n} x_i w_i$
  - c) Poly. time

Only one can hold for P-hardness

Can not hold together (NP-hardness)

this creates „almost" optimal mechanisms. (The DSIC is nice to keep)

Consider greedy allocation X_G, sort the bidders so that $\frac{b_i}{w_i} \ge \frac{b_{i+1}}{w_{i+1}}$

1) we select winners in backpack until they fit
2) declare them or the highest bidder as winners, whatever gives larger soc. surplus

X_G is monotone $\Rightarrow$ DSIC

Thm: Assuming truthful bids X_G gives $\ge \frac{1}{2}$ of max. soc. surplus

Dk:

Fractional variant:

- $\forall$ bidder can be allocated with fraction $\alpha_i \in \langle 0, 1 \rangle$
- values: αv_i, sizes: αw_i
- for this, we do step 1) and place the first losing bidder fractionally.
- this algorithm gives optimum in the fractional setting
  - assume we selected k winners with $\alpha_1 \ldots \alpha_n$ ($\alpha_1 = \ldots = \alpha_k = 1$) — first k fully in
  - let suppose for $\frac{1}{2}$ $\exists$ better solution $\beta_1 \ldots \beta_n$

$\Rightarrow \exists i \ge k : \beta_i > \alpha_i$ AND $\exists j < i : \beta_j < \alpha_j$

because otherwise backpack overfit — i need to decrease the one which is clearly in backpack

$\Rightarrow (\beta_i - \alpha_i) v_i > (\alpha_j - \beta_j) \cdot v_j$ AND $(\beta_i - \alpha_i) w_i \le (\alpha_j - \beta_j) w_i$

$\Rightarrow \frac{v_i}{w_i} > \frac{v_j}{w_j}$ $\to$ but this is contradiction, since $v_i = b_i$ and $i > j$

Soc. surplus obtained by X_G is $\geq MAX\left\{\sum_{i=1}^{k-1} v_i, v_k\right\} \geq$

$$\frac{1}{2}\sum_{i=1}^{k} \alpha_i v_i = \frac{1}{2} OPT_{FRAC} \geq \frac{1}{2} OPT. \quad \boxtimes$$

$max(\delta_1, \delta_2) \geq \frac{1}{2}(\delta_1 + \delta_2)$ – triviálně