Bayesian mode consists of:

1) Single par. mechanism (X, p)

- n bidders, with valuations $v_1 - v_n$
- X = allocation rule
- p = payment rule
- feasible outcomes X

2) Valuations $v_1 - v_n$ are drawn from independent prob. distr $F_1 - F_n$ with densities $f_1 - f_n$ with support on $\langle 0, v_{max} \rangle$

$$F_i(z) = P_r[v_i \leq z]$$

$$f_i(x) = \frac{\partial}{\partial x} F_i(x), \quad F_i(x) = \int_0^x f_i(z)\, dz$$

3) $F_1 - F_n$ are known to the seller

- we try to maximize $\mathbb{E}_{v \sim (F_1 - F_n)}\left[\sum_i p_i(v)\right]$ = expected revenue

- $\mathbb{E}_{v \sim F}[X(v)] = \int_0^{v_{max}} X(v) f(v)\, dv$

Example 1: (single-item, 1 bidder)

- consider uniform prob. dist. F on $\langle 0,1 \rangle$ $\quad F(x) = x \;\; \forall x \in \langle 0, 1 \rangle$

$\Rightarrow$ exp. revenue is $r \cdot (1 - F(r)) = r \cdot (1 - r)$ $\Rightarrow$ this gives maximum $1/4$ for $r = 1/2$

Example 2: (single-item, 2 bidders)

- we can apply vickre's auction $\Rightarrow$ exp. revenue is $1/3$
- $\exists$ better auction that gives $5/12$.

Thm:

In $\forall$ single-par env. $\forall$ DSIC (X, p) with prob. distr. $F_1 - F_n$ with densities $f_1 - f_n$ if $F = F_1 \times \dots \times F_n$, then $\mathbb{E}_{v \sim F}\left[\sum_i p_i(v)\right] = \mathbb{E}_{v \sim F}\left[\sum_i \varphi_i(v_i) \cdot x_i(v)\right]$

where $\underbrace{\varphi_i(v_i)}_{\text{virtual valuation}} = \underbrace{v_i}_{\text{valuation}} - \underbrace{\frac{1 - F_i(v_i)}{f_i(v_i)}}_{\text{inevitable loss for not knowing } v_i \text{ exactly.}}$

this would be social surplus without φ

Therefore it is called virtual social surplus.

- can be negative (if F uniform on $\langle 0, 1 \rangle \Rightarrow \varphi_i(v_i) = v_i - \frac{1 - v_i}{1} = 2v_i - 1 \in \langle -1, 1 \rangle$ then.

Ok: (x,p) is DSIC $\Rightarrow b_i = v_i$ „bids are valuations"

$\Rightarrow$ myerson's lemma $\Rightarrow p_i(v) = \int_0^v z \cdot \frac{\partial}{\partial z} x_i(z, v_{-i})\, dz$

fix i, v_{-i}:

$$\mathbb{E}_{v_i \sim F_i}\left[p_i(v_i, v_{-i})\right] = \int_0^{v_{max}} p_i(v_i, v_{-i}) \cdot f_i(v_i)\, dv_i = \int_0^{v_{max}} \int_0^{v_i} z \frac{\partial}{\partial z} x_i(z, v_{-i}) \cdot f_i(v_i)\, dz\, dv_i =$$

definition of $\mathbb{E}$

Fubini thm

$$= \int_0^{v_{max}} \underbrace{\left(\int_z^{v_{max}} f_i(v_i)\, dv_i \right)}_{1 - F_i(x)} \cdot z \frac{\partial}{\partial z} x_i(z, v_{-i})\, dz = \int_0^{v_{max}} \left(1 - F_i(z)\right) \cdot z \cdot \frac{\partial}{\partial z} \cdot x_i(z, v_{-i})\, dz$$

per partes $\int f g' = f \cdot g - \int f' \cdot g$: $f(z) = (1 - F_i(z))$
$g(z)\ \frac{\partial}{\partial z} x_i(z, v_{-i})$

$$= \underbrace{\left[(1 - F_i(z)) \cdot z \cdot x_i(z, v_{-i}) \right]_0^{v_{max}}}_{=0} - \int_0^{v_{max}} \left(1 - F_i(z) - z \cdot f_i(z)\right) \cdot x_i(z, v_{-i})\, dz$$

definition of $\mathbb{E}$

$$= \int_0^{v_{max}} \left(z - \frac{1 - F_i(z)}{f_i(z)} \right) \cdot f_i(z) \cdot x_i(z, v_{-i})\, dz = \mathbb{E}_{v_i \sim F_i}\left[\varphi_i(v_i) \cdot x_i(v_i, v_{-i}) \right]$$

only rewriting

$$\Rightarrow \forall i\ \forall v_{-i}: \mathbb{E}_{v_i \sim F_i}\left[p_i(v_i, v_{-i})\right] = \mathbb{E}_{v_i \sim F_i}\left[\varphi_i(v_i)\, x_i(v_i, v_{-i})\right]$$

integration over all v_{-i}

$$\Rightarrow \underline{\underline{\mathbb{E}_{v \sim F}\left[p_i(v)\right]}} = \mathbb{E}_{v \sim F}\left[\varphi_i(v_i) \cdot x_i(v)\right]$$

$$\Rightarrow \mathbb{E}_{v \sim F}\left[\sum_i p_i(v)\right] = \sum_i \mathbb{E}_{v \sim F}\left[p_i(v)\right] = \sum_i \mathbb{E}_{v \sim F}\left[\varphi_i(v_i)\, x_i(v)\right] = \underline{\underline{\mathbb{E}_{v \sim F}\left[\sum_i \varphi_i(v_i) \cdot x_i(v)\right]}}$$

Lets apply the thm. Consider single-item auction, $F = F_1 = \dots = F_n \Rightarrow f = f_1 = \dots = f_n$
Assume that F is regular (φ is strictly increasing) $\Rightarrow \varphi = \varphi_1 = \dots = \varphi_n$
(true for F uniform on $\langle 0, 1 \rangle$)

Vichrey's auction with reverse price m → This is eBay auction!

- winner is the highest bidder, but if bids are $< m$, then there is no winner. Otherwise we get negative $\varphi_i(v_i)$
- winner (if any) pays 2nd highest bid or m, whatever is larger.

Under our assumptions, vichrey's auction with reserve price $\varphi^{-1}(0)$ maximizes revenue. → since F is regular, only one image exists.

Dk: We want to maximize expected virtual valuation

$$X = \{(x_1 \dots x_n) \in \{0,1\}^n : \sum_i x_i \leq 1\} \Rightarrow \text{winner should have highest } \varphi_i(v_i)$$

- if $\varphi_1(v_1), \dots, \varphi_n(v_n) < 0 \Rightarrow \nexists$ winner
- meyrson's lemma gives the payment rule - pay $m = \varphi^{-1}(0)$ or 2nd highest $\varphi_i(v_i)$

$\Rightarrow$ Vichrey's auction with reserve price $\varphi^{-1}(0)$. $\boxed{X}$

→ allocation rule is monotone (if F regular)