Proof of minimax:

- we want to use LP to compute Worst Case Optimal strategy.

$\sum_i^m x_i = 1,\ x_1 - x_m \geq 0,\ MAX\ \beta(x)$ --> which is not linear.

Let x be fixed. ——> here it must be fixed to make problem linear

We construct LP that computes $\beta(x)$.

$(P) \xleftrightarrow{\text{duality}} (D)$

(P)

Variables: $y_1 \dots y_n$

Obj. function: $MIN\ x^T M y$

Constraints: $\sum_j^n y_j = 1$

$y_1 - y_n \geq 0$

(D)

Variables: x_0 <-- Because we have just one constraint.

Obj. function: $MAX\ x_0$ -> inverse of MIN in P

Constraints: $\mathbb{1} x_0 \leq M^T x$ -> $\mathbb{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$x_0 \in \mathbb{R}$

D'

Variables: $x_0, x_1 - x_m$

Obj. function: $MAX\ x_0$

Constraints: $\mathbb{1} x_0 - M^T x \leq 0$

$\sum_i^m x_i = 1$

$x_0 \in \mathbb{R},\ x_1 - x_m \geq 0$

-> D' finds $x \in S_1$ that maximizes $\beta(x)$

↳ computes WSO $\bar{x}$

↳ analogously for x and wso of $\alpha(y)$

P'

Variables: $y_0 - y_n$

Obj. function: $MIN\ y_0$

Constraints: $\mathbb{1} y_0 - My \geq 0$

$\sum_j^n y_j = 1$

$y_0 \in \mathbb{R},\ y_1 - y_n \geq 0$

Actually: P' and D' are dual.

Duality thm: => $\forall$ wso $\bar{x}, \bar{y}$, $\alpha(\bar{y}) = \beta(\bar{x})$

Lemma 2.20.c => $(\bar{x}, \bar{y})$ is NE. $\boxtimes$

NE in bimatrix games

→ General name for 2-player games

for example: Prisoner's dilemma

By brute force:

$A_1 = \{1 \dots m\}$, $A_2 = \{1 \dots n\}$ $(A_2 = \{m+1 \dots n\})$

$s = (s_1, s_2) \in S$, $x_i = s_1(i)$, $x_j = s_2(j)$

$M, N \in \mathbb{R}^{n \times m}$, $(M)_{ij} = u_1(i,j)$, $(N)_{ij} = u_2(i,j)$

$U_1(s) = x^T M y$

$u_2(s) = x^T N y$

Best response condition:

For any normal-form games $G(P, A, u)$ of n players, $\forall S = (s_1 \dots s_n) \in S$, $\forall i \in P$,

s_i is best response to s_{-i} $\iff$ $\forall a_i \in \underbrace{\text{support}(s_i)}$: → $s_i = \{a_i \in A_i : s_i(a_i) > 0\}$

a_i is best response to s_{-i}

Dk:

$\Leftarrow$) Assume $\forall a_i \in \text{Supp}(s_i)$, a_i is best response to s_{-i}.

Want to show that s_i is best response to s_{-i}

Let $s_i' \in S_i$, $u_i(s) \overset{\text{linearity}}{=} \sum_{a_i \in \text{supp}(s_i)} s_i(a_i) \cdot \underbrace{u_i(a_i, s_{-i})}_{\geq u_i(s_i', s_{-i}) \text{ because } \forall a_i \in \text{support} \dots} \geq u_i(s_i', s_{-i}) \cdot \overbrace{\sum_{a_i \in \text{supp}(s_i)} s_i(a_i)}^{1} = u_i(s_i', s_{-i})$

therefore S_i is best response.

$\Rightarrow$ Let s_i be a best response to s_{-i}. Suppose $\exists \bar{a}_i \in \text{Supp}(s_i)$ that is not best response to s_{-i}.

$\Rightarrow \exists s_i' \in S_i : u_i(s_i', s_{-i}) > u_i(\bar{a}_i, s_{-i})$

s_i is best response $\Rightarrow \exists \hat{a}_i \in \text{supp}(s_i) : u_1(\hat{a}_i, s_{-i}) > u_1(\bar{a}_i, s_{-i})$

→ if there is better utility, there also must be better action $(\hat{a}_i)$

- Let s_i^* be strategy corresponding to s_i but where-ever instead of $\bar{a}_i$ we play $\hat{a}_i$. → But then there is a better strategy than s_i. (s_i^*)

s_i^* gives me better utility: $u_i(s) = \sum_{a_i \in \text{supp}(s_i)} s_i(a_i) \cdot u_i(a_i, s_{-i}) < \sum_{a_i \in \text{supp}(s_i^*)} s_i^*(a_i) \cdot u_i(a_i, s_{-i}^*)$

→ because of better action in s_i^*.

Therefore s_i is not the best response. $\boxtimes$

x is best response to $y \iff \forall i \in A_1 : x_i > 0 \implies \underbrace{(M)_i y = MAX \{(M)_h y : h \in A_1\}}_{u_1(i, s_2) \geq u_1(h, s_2)\ \forall h \in A_1}$

(*) $\quad i \in supp(s_1)$

y is best response to $x \iff \forall j \in A_2 : y_j > 0 \implies (N^T)_j x = MAX \{(N^T)_h x : h \in A_2\}$

A bimatrix game is non-generate if for $\forall$ strategy with support of size k, there $\exists \leq k$ pure best responses.

Let $I \subseteq A_1$ and $J \subseteq A_2$ be our supports.

- Is there NE (x,y) with $supp(x) = I$ and $supp(y) = J$?
  - to decide, it suffices to solve system of linear equations.

Variables: $x_i : i \in I$, $y_j : j \in J$, u, v $\to$ in total $|I| + |J| + 2$ vars)

Equalities: $\sum_i^m x_i = 1$, $\sum_j^n y_j = 1$, $\forall i \in I : (M)_i y = u$, $\forall j \in J : (N^T)_j x = v$

$\hookrightarrow$ in total $|I| + |J| + 2$ equalities

If (x,y) is a solution with $x, y \geq 0$ and $u = MAX \{(M)_h y : h \in A_1\}$

$v = MIN \{(N^T)_h x : h \in A_2\}$

Then (x,y) is NE. (By best response con)

Brute force algorithm:

Input: Bimatrix game G, non-degenerate

Output: All NE:

Go over all $I \subseteq A_1, J \subseteq A_2$, given I and J solve the system of EQ, possibly return NE.

If G is non-degenemte, then it suffices to check $|I| = |J|$.

$\hookrightarrow$ if (x,y) is NE with $|supp(x)| < |supp(y)|$

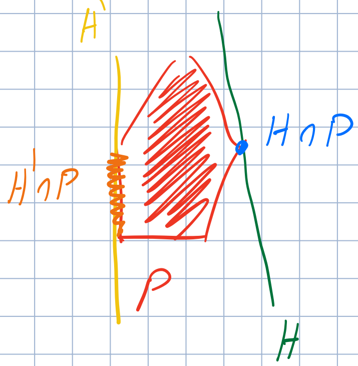
contradiction with non-degenerate games. I have more responses to x then $|supp(x)|$

$\hookrightarrow$ so if $m = n$, then we need $\lesssim 4^n$ steps. $\hookrightarrow$ which is terrible, but in P.

Polyhedron P:

FACE of P is $P \cap H$ where P is contained in a halfspace determined by P.

hyperplane



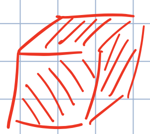
empty set can be face too.

face of dim $= 0 =$ vertex

$1 =$ edge

$d-1 =$ facet

P is SIMPLE if $\forall$ vertex is in $\leq d$ facets.



→ is simple: every vertex is on three sides.