Best response polyhedron:

For 1 is $\overline{P} = \{(x,v) \in \mathbb{R}^{m+1} : x \geq 0, \mathbb{1}^T x = 1, N^T x \leq \mathbb{1} v\}$

For 2 is $\overline{Q} = \{(y,u) \in \mathbb{R}^{n+1} : y \geq 0, \mathbb{1}^T y = 1, M y \leq \mathbb{1} u\}$

u is a upperbound on $u_1(s)$: $u_1(s) = x^T \underbrace{My}_{\leq \mathbb{1}u} \leq x^T \mathbb{1} u = u$

(x,v) of $\overline{P}$ has label $i \in A_1 \cup A_2$ if:

either $(i \in A_1$ and $x_i = 0)$ or $(i \in A_2$ and $(N^T)_i x = v)$

(y,u) of $\overline{Q}$ has label $i \in A_1 \cup A_2$ if:

either $(i \in A_1$ and $(M)_i y = u)$ or $(i \in A_2$ and $y_i = 0)$

– some vertices can have more labels, edge have only one.

Let $s = (s_1, s_2)$ with vectors x, y is NE $\iff$ $(x, u_1(s)) \times (y, u_2(s)) \in \overline{P} \times \overline{Q}$ is completely labeled.

$\forall i \in A_1 \cup A_2$ appears as a label of $(x, u_1(s))$ or $(y, u_2(s))$

Proof:

$\Rightarrow$ Contradiction:

Suppose $i \in A_1 \cup A_2$ is missing as a label.

WLOG – $i \in A_1$. (x,v) does not have label i. $\Rightarrow x_i > 0$ (is in support)

(y,u) does not have label i. $\Rightarrow (M)_i y < u$ (this is not best response)

Best response condition $\Rightarrow$ x is not best response y $\Rightarrow$ (x,y) is not NE. $\boxtimes$

$\Leftarrow$: Completely labeled $\to$ everything in $supp(x), supp(y)$ is best response.

Best response condition $\Rightarrow (x,y)$ is NE. $\boxtimes$

Assume M, N are non-negative and have no zero column.

We transform $\overline{P}$ and $\overline{Q}$ by substituting $\underbrace{(x,v) \mapsto \frac{x}{v}, (y,u) \mapsto \frac{y}{u}}_{\text{projective transformation}}$ } P, Q are bounded and $\dim(P) = m$, $\dim(Q) = n$

Normalized best response polytope

For 1 is $P = \{x \in \mathbb{R}^m : x \geq 0, N^T x \leq \mathbb{1}\}$

For 2 is $Q = \{y \in \mathbb{R}^n : y \geq 0, My \leq \mathbb{1}\}$

they are not mixed-strategy, they don't sum up to 1.

But $\frac{x}{\mathbb{1}^T x}$ is the mixed-strategy profile.

there is bijection between $\overline{P}$ and $P \setminus \{0\}$

$\overline{Q}$ and $Q \setminus \{0\}$

that preserves incidences $\Rightarrow$

labels are preserved.

If G is non-degenerate, then $\forall x \in P$ has $\leq m$ labels
Q has $\leq n$ labels

If $|supp(x)| = k \Rightarrow$ there are $m-k$ labels from A_1 at x
If x has $> m$ labels $\Rightarrow x$ has $> k$ labels from A_2.
$\Rightarrow \exists > k$ pure best resp. to $x \Rightarrow G$ is degenerate $\lightning$
$\Leftrightarrow (x,y)$ is completely labeled, then it is a pair of vertices.
$\Rightarrow P, Q$ are simple.

Lemke-Howson algorithm:

$\longrightarrow$ finding only single one, not all

Dropping a label $\ell \in A_1 \cup A_2$ at $x \in P$
= walking along the edge that is missing ℓ to a new vertex, where we pick up a new label ℓ'.

Pseudocode:

Start at $(0,0) \in P \times Q$, we pick $k \in A_1 \cup A_2$.
$\ell \leftarrow k$ $\longrightarrow$ putting k in the support

Smart!

while (true)
  in P, drop ℓ, $x \leftarrow$ new vertex
    $\ell \leftarrow$ picked up label
  switch to Q, if $\ell = k \rightarrow$ STOP, NE
  In Q, drop ℓ, $y \leftarrow$ new vertex
    $\ell \leftarrow$ picked up label
  switch to P, if $\ell = k \rightarrow$ STOP, NE

Correctness of Lemke-Howson:

every label appears except possibly k

Non-degenerate G, $k \in A_1 \cup A_2$, lets define configuration graph $\mathcal{G} = (V,E)$,

$V = \{(x,y) \in P \times Q : (x,y) \text{ is } k\text{-almost completely labeled}\}$

$E = \{(x,y),(x',y') : x = x' \text{ and } yy' \text{ is edge of } Q \text{ or } y = y' \text{ and } xx' \text{ is edge of } P\}$

$\mathcal{G}$ is finite, $\forall e \in E: \deg(e) \in \{1,2\}$

$\Leftrightarrow$ it is therefore bunch of paths and cycles

If (x,y) is completely labeled $\Rightarrow$ we can only drop k $\rightarrow$ degree 1.

If (x,y) is missing k $\Rightarrow$ (x,y) has same $l \neq k$ twice $\Rightarrow$ I can drop one $\rightarrow$ degree 2

We start at completely labeled $(0,0)$ $\rightarrow$ it has degree 1 and therefore is an endpoint in the path.

In general step, we traverse edge of $\mathcal{G}$.

We can never go back (loop), because path has edges from P and Q repeatedly: $\underset{P}{—}\circ\underset{Q}{—}\circ\underset{P}{—}\circ\underset{Q}{—}$ and since I switch P and Q every step,

I can not loop $\Rightarrow$ we have completely labeled pair $\Rightarrow$ NE.

Check $(x^*, y^*) \neq (0, y^*), (x^*, 0)$ $\rightarrow$ proof left for homework.