What will be in the exam

Will not be

Introduction to Natural Language Processing I [Statistické metody zpracování přirozených jazyků I] (NPFL067)

https://ufal.mff.cuni.cz/courses/npfl067

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Intro to Stat. NLP I

- Instructors: Jan Hajič / Jindřich Helcl
  - ÚFAL MFF UK, office: 420, office hours: J. Hajic: Mon 10:30-11:00
  - preferred contact: **{hajic,helcl}@ufal.mff.cuni.cz**
- Room & time:
  - lecture: room S1, Mon 10:40-12:10
  - Seminar/practice [cvičení] not used for in-person class (room S8, 15:40-17:20)
  - Sep 30, 2024 – Jan 10, 2025, main final written exam date: Jan 13, 2025
- Inverted course:
  - you watch the lectures at your leisure
  - Interactive part at seminars (in/person, online for those with travel difficulties)

Textbooks you need

- Manning, C. D., Schütze, H.:
  - *Foundations of Statistical Natural Language Processing*. The MIT Press. 1999. ISBN 0-262-13360-1. **[required]**
- Jurafsky, D., Martin, J.H.:
  - *Speech and Language Processing.* Prentice-Hall. 2000. ISBN 0-13-095069-6 and later editions. **[recommended].**

Other reading

- Charniak, E:
  - *Statistical Language Learning*. The MIT Press. 1996. ISBN 0-262-53141-0.
- Cover, T. M., Thomas, J. A.:
  - *Elements of Information Theory*. Wiley. 1991. ISBN 0-471-06259-6.
- Jelinek, F.:
  - *Statistical Methods for Speech Recognition*. The MIT Press. 1998. ISBN 0-262-10066-5
- Proceedings of major conferences (ACL Anthology)
  - ACL (Assoc. of Computational Linguistics)
  - EACL/NAACL/IJCNLP (European/American/Asian Chapter of ACL)
  - EMNLP (Empirical Methods in NLP), CONLL
  - COLING (Intl. Committee of Computational Linguistics)
- arXiv, journals: Computational Linguistics, TACL

Course requirements

- Grade components: requirements & weights:
  - Homeworks (2): 1/3
  - Final Exam: 1/3
- Exam:
  - approx. 4-5 questions:
    - mostly explanatory answers (1/4 page or so),
    - algorithms
    - only a few multiple choice questions

Homeworks

- Homeworks:
  - Entropy, Language Modeling; Word Classes
- Organization
    - (little) paper-and-pencil exercises, lot of programming
    - turning-in mechanism: see the web
    - no plagiarism!
- Deadlines
  - See the course web (Nov. 30, 2024; Feb 29, 2025)
  - Late penalty: 5% of grade (0-100) per day (max. 50%)

Course segments

- Intro & Probability & Information Theory
  - The very basics: definitions, formulas, examples.
- Language Modeling
  - n-gram models, parameter estimation → very easy computation, almost „you can see"
  - smoothing (EM algorithm)
- Words and the Lexicon
  - word classes, mutual information, bit of lexicography
- Hidden Markov Models
  - background, algorithms, parameter estimation
- Decision Trees in NLP

NLP: The Main Issues

- Why is NLP difficult?
  - many “words”, many “phenomena” --> many “rules”
    - **OED: 400k words; Finnish lexicon (of forms): ~2 . 10^7**
    - **sentences, clauses, phrases, constituents, coordination, negation, imperatives/questions, inflections, parts of speech, pronunciation, topic/focus, and much more!**
  - irregularity (exceptions, exceptions to the exceptions, ...)
    - **potato -> potato es (tomato, hero,...); photo -> photo s,** and even: both **mango -> mango s** or **-> mango es**
    - Adjective / Noun order: **new book, electrical engineering, general regulations, flower garden, garden flower, ...:** but **Governor General**

Difficulties in NLP (cont.)

- ambiguity
  - **books: NOUN** or **VERB?**
    - you **need** many books *vs.* she **books** her flights online
  - **No left turn weekdays 4-6 pm / except transit vehicles (***Charles Street at Cold Spring***)**
    - when may transit vehicles turn: Always? Never?
  - **Thank you for not smoking, drinking, eating or playing radios without earphones.** *(MTA bus)*
    - Thank you for not eating without earphones??
    - or even: Thank you for ~~not~~ drinking without earphones!?
  - **My neighbor's hat was taken by wind. He tried to catch it.**
    - ...catch the wind *or* ...catch the hat ?

(Categorical) Rules or Statistics?

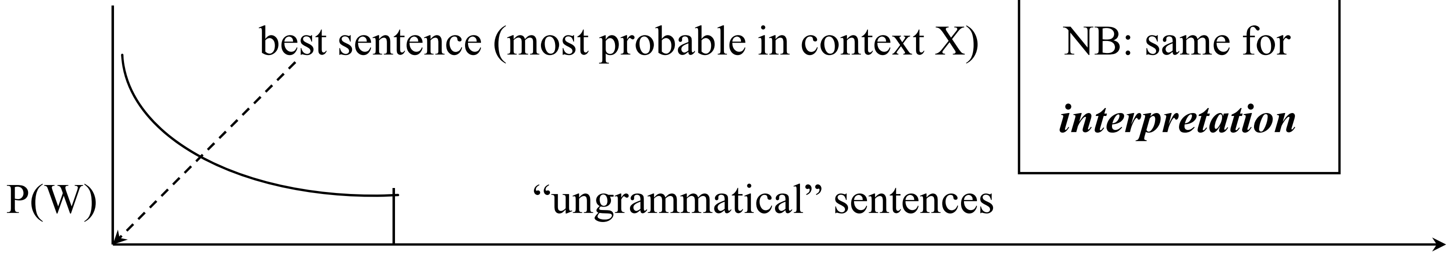
- Preferences:
  - clear cases: context clues: she books --> books is a verb
    - rule: if an ambiguous word (verb/nonverb) is preceded by a matching personal pronoun -> word is a verb
  - less clear cases: pronoun reference
    - she/he/it refers to the most recent noun or pronoun (?) (but maybe we can specify exceptions)
  - selectional:
    - catching hat >> catching wind (but why not?)
  - semantic:
    - never thank for drinking in a bus! (but what about the earphones?)

Solutions

- Don’t guess if you know:
  - **morphology (inflections)**
  - **lexicons (lists of words)**
  - **unambiguous names**
  - **perhaps some (really) fixed phrases**
  - **syntactic rules?**
- Use statistics (based on real-world data) for preferences (only?)
  - **No doubt: but this is the big question!**
  - **Statistics ~ Machine learning (Neural Networks / LLMs / …)**

Statistical NLP

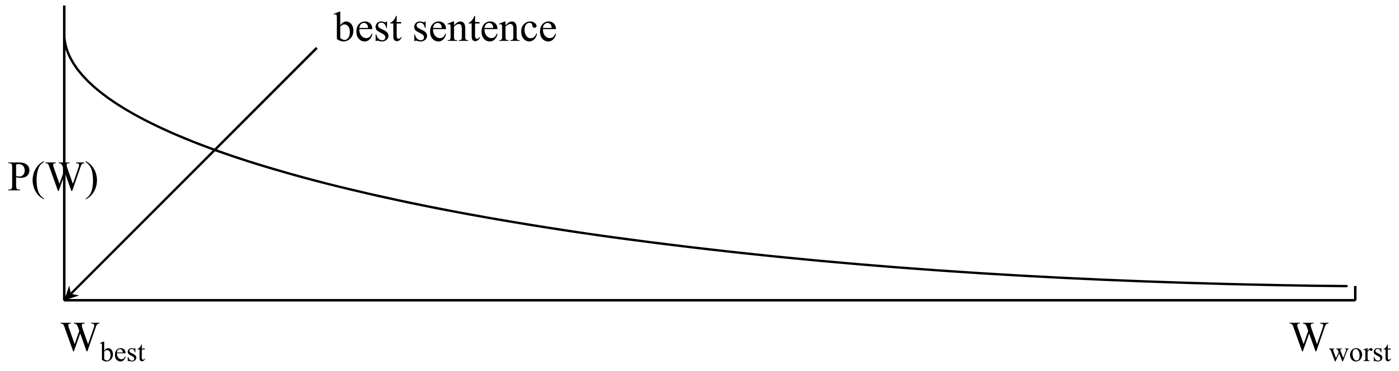
- Imagine:
  - Each sentence $W = \{ w_1, w_2, ..., w_n \}$ gets a probability $P(W|X)$ in a context X (think of it in the intuitive sense for now)
  - For every possible context X, sort all the imaginable sentences W according to $P(W|X)$:
  - Ideal situation:



NB: same for ***interpretation***

Real World Situation

- Unable to specify set of grammatical sentences today using fixed "categorical" rules (maybe never, cf. arguments in MS)
- Use statistical "model" based on **<u>REAL WORLD DATA</u>** and care about the best sentence only (disregarding the "grammaticality" issue)



Probability

Experiments & Sample Spaces

- Experiment, process, test, ...
- Set of possible basic outcomes: sample space Ω
  - coin toss (Ω = {head,tail}), die (Ω = {1..6})
  - yes/no opinion poll, quality test (bad/good) (Ω = {0,1})
  - lottery ($|\Omega| \cong 10^7 .. 10^{12}$)
  - # of traffic accidents somewhere per year (Ω = N)
  - spelling errors ($\Omega = Z^*$), where Z is an alphabet, and Z^* is a set of possible strings over such and alphabet
  - missing word ($|\Omega| \cong$ vocabulary size)

Events

- Event A is a set of basic outcomes
- Usually $A \subset \Omega$, and all $A \in 2^{\Omega}$ (the event space)
  - Ω is then the certain event, $\emptyset$ is the impossible event
- Example:
  - experiment: three times coin toss
    - **Ω = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}**
  - count cases with exactly two tails: then
    - **A = {HTT, THT, TTH}**
  - all heads:
    - **A = {HHH}**

Probability

- Repeat experiment many times, record how many times a given event A occurred ("count" c_1).
- Do this whole series many times; remember all c_is.
- Observation: if repeated really many times, the ratios of $\mathbf{c_i/T_i}$ (where T_i is the number of experiments run in the *i-th* series) are close to some (unknown but) **constant** value.
- Call this constant a ***probability of A***. Notation: **p(A)**

Estimating probability

- Remember: ... close to an *unknown* constant.
- We can only estimate it:
  - from a single series (typical case, as mostly the outcome of a series is given to us and we cannot repeat the experiment), set

$$p(A) = c_1/T_1.$$

  - otherwise, take the weighted average of all c_i/T_i (or, if the data allows, simply look at the set of series as if it is a single long series).
- This is the **best** estimate.

Example

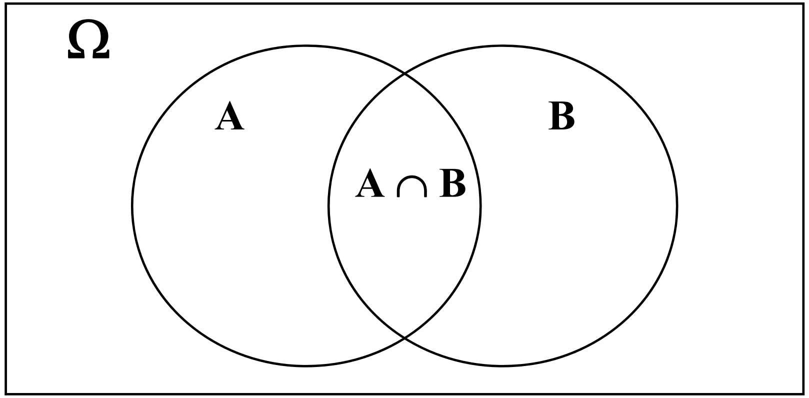
- Recall our example:
  - experiment: three times coin toss
    - Ω = **{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}**
  - count cases with exactly two tails: A = **{HTT, THT, TTH}**
- Run an experiment 1000 times (i.e. 3000 tosses)
- Counted: 386 cases with two tails (**HTT, THT,** or **TTH**)
- estimate: p(A) = 386 / 1000 = .386
- Run again: 373, 399, 382, 355, 372, 406, 359
  - p(A) = .379 (weighted average) or simply 3032 / 8000
- *Uniform* distribution assumption: p(A) = 3/8 = .375

Basic Properties

- Basic properties:
  - p: $2^{\Omega} \rightarrow [0,1]$
  - $p(\Omega) = 1$
  - Disjoint events: $p(\cup A_i) = \sum_i p(A_i)$
- [NB: ***axiomatic definition*** of probability: take the above three conditions as axioms]
- Immediate consequences:
  - $p(\emptyset) = 0$, $p(\overline{A}) = 1 - p(A)$, $A \subseteq B \Rightarrow p(A) \leq p(B)$
  - $\sum_{a \in \Omega} p(a) = 1$

Joint and Conditional Probability

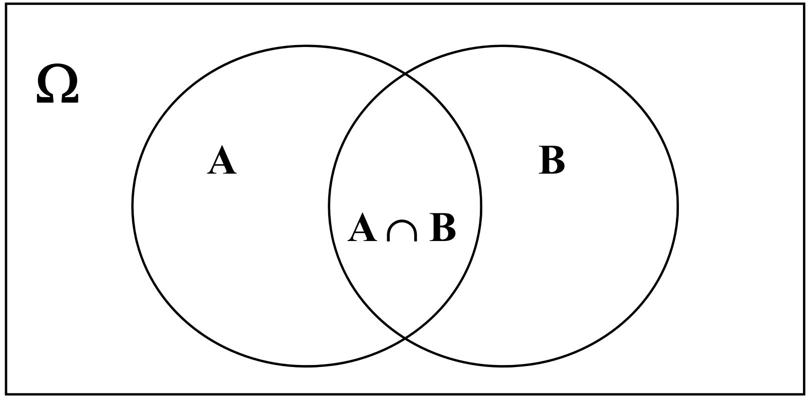
- $p(A,B) = p(A \cap B)$
- $p(A|B) = p(A,B) / p(B)$ → jde opět o distribuční
  – Estimating form counts:
    - $\mathbf{p(A|B) = p(A,B) / p(B) = (c(A \cap B) / T) / (c(B) / T) = c(A \cap B) / c(B)}$



Bayes Rule

- $p(A,B) = p(B,A)$ since $p(A \cap B) = p(B \cap A)$
  - therefore: $p(A|B)\ p(B) = p(B|A)\ p(A)$, and therefore

$$p(A|B) = p(B|A)\ p(A) \,/\, p(B) \quad !$$



Independence

- Can we compute p(A,B) from p(A) and p(B)?
- Recall from previous foil:

$$p(A|B) = p(B|A)\ p(A) / p(B)$$

$$p(A|B)\ p(B) = p(B|A)\ p(A)$$

$$p(A,B) = p(B|A)\ p(A)$$

... we're almost there: how p(B|A) relates to p(B)?

  – p(B|A) = P(B) iff A and B are **<u>independent</u>**

- Example: two coin tosses, weather today and weather on March 4th 1789;
- Any two events for which p(B|A) = P(B)!

Chain Rule

→ we later use it with limited history

$$p(A_1, A_2, A_3, A_4, \ldots, A_n) =$$

$$p(A_1|A_2,A_3,A_4,\ldots,A_n) \times p(A_2|A_3,A_4,\ldots,A_n) \times$$
$$\times\, p(A_3|A_4,\ldots,A_n) \times \ \ldots\ \ p(A_{n-1}|A_n) \times p(A_n)$$

!

- this is a direct consequence of the Bayes rule.

The Golden Rule
(of Classic Statistical NLP)

- Interested in an event A given B (when it is not easy or practical or desirable to estimate p(A|B)):
- take Bayes rule, max over all As:
- $\text{argmax}_A\ p(A|B) = \text{argmax}_A\ p(B|A)\ .\ p(A)\ /\ p(B) =$

$$\text{argmax}_A\ p(B|A)\ p(A)\ !$$

- ... as p(B) is constant when changing As

typically A is the output

Random Variable

- is a function $X: \Omega \rightarrow Q$
  - in general: $Q = R^n$, typically R
  - easier to handle real numbers than real-world events
- random variable is *discrete* if Q is countable (i.e. also if finite)
- Example: *die*: natural "numbering" [1,6], *coin*: {0,1}
- Probability distribution:
  - $p_X(x) = p(X=x) =_{df} p(A_x)$ where $A_x = \{a \in \Omega : X(a) = x\}$
  - often just p(x) if it is clear from context what X is

Expectation
Joint and Conditional Distributions

- is a mean of a random variable (weighted average)
  - $E(X) = \sum_{x \in X(\Omega)} x \cdot p_X(x)$
- Example: one six-sided die: 3.5, two dice (sum) 7
- Joint and Conditional distribution rules:
  - analogous to probability of events
- Bayes: $p_{X|Y}(x,y) =_{\text{notation}} p_{XY}(x|y) =_{\text{even simpler notation}}$

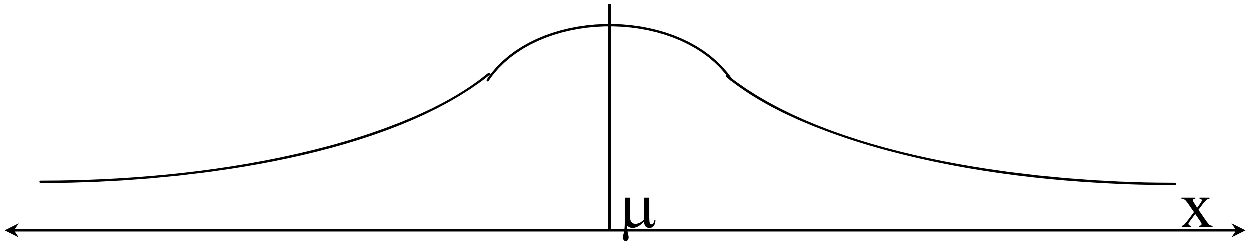
  $$\mathbf{p(x|y) = p(y|x) \,.\, p(x) \,/\, p(y)}$$
- Chain rule: $\mathbf{p(w,x,y,z) = p(z).p(y|z).p(x|y,z).p(w|x,y,z)}$

Standard distributions

- Binomial (discrete)
  - outcome: 0 or 1 (thus: *bi*nomial)
  - make *n* trials
  - interested in the (probability of) number of successes *r*
- Must be careful: it's not uniform!
- $p_b(r|n) = \binom{n}{r} / 2^n$ (for equally likely outcome)
- $\binom{n}{r}$ counts how many possibilities there are for choosing r objects out of n; $= n! / ((n-r)!\, r!)$

Continuous Distributions

- The normal distribution ("Gaussian")
- $p_{norm}(x|\mu,\sigma) = e^{-(x-\mu)^2/(2\sigma^2)}/\sigma\sqrt{2\pi}$
- where:
  - μ is the mean (x-coordinate of the peak) (0)
  - σ is the standard deviation (1)



- other: hyperbolic, t

Essential Information Theory

The Notion of Entropy

- Entropy ~ “chaos”, fuzziness, opposite of order, ...
  - you know it:
    - **it is much easier to create “mess” than to tidy things up...**
- Comes from physics:
  - Entropy does not go down unless energy is applied
- Measure of ***uncertainty:***
  - if low... low uncertainty; the higher the entropy, the higher uncertainty, but the higher “surprise” (information) we can get out of an experiment

The Formula

- Let $p_X(x)$ be a distribution of random variable X
- Basic outcomes (alphabet) Ω

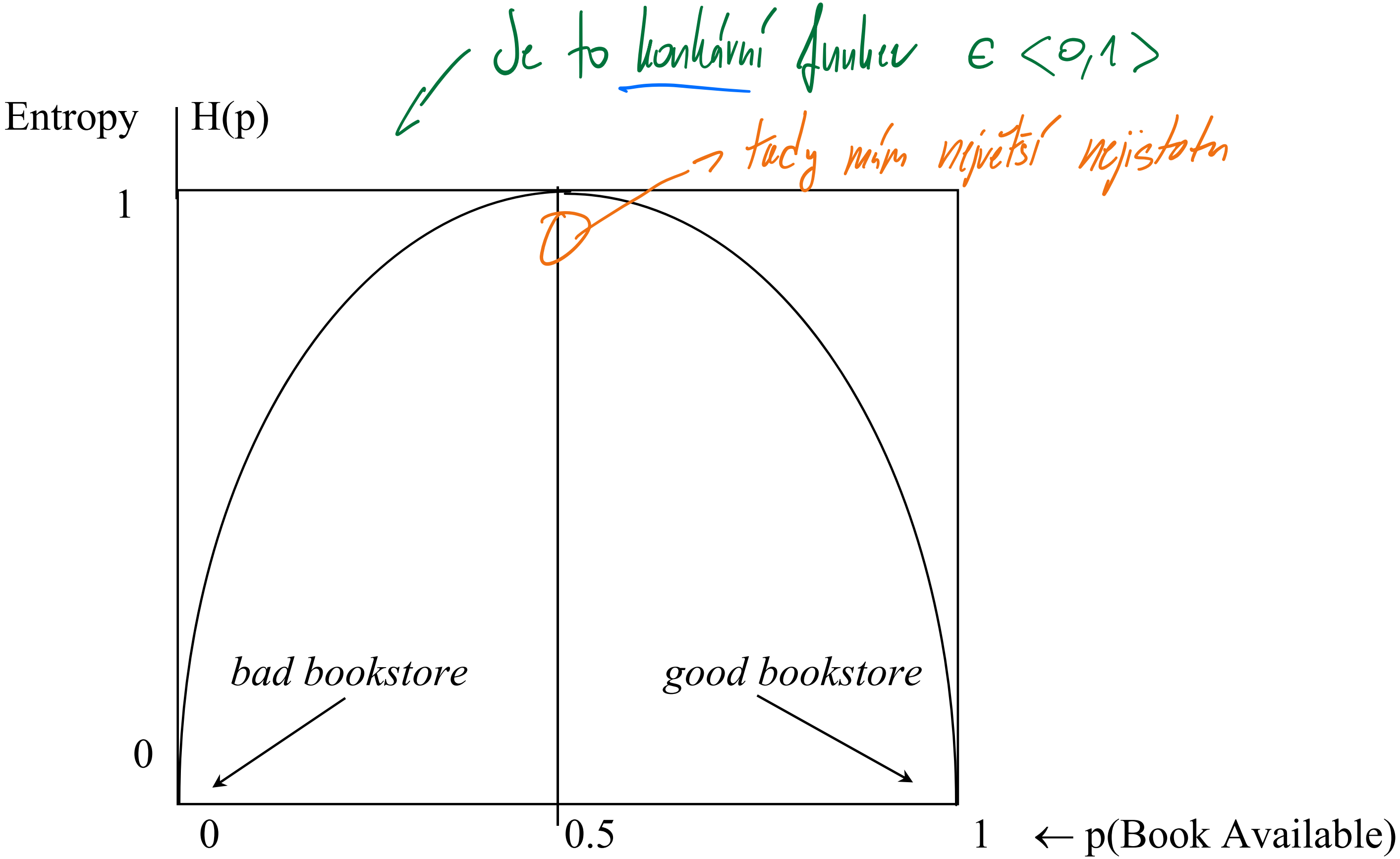
$$H(X) = - \sum_{x \in \Omega} p(x) \log_2 p(x) \quad !$$

- Unit: bits ($\log_{10}$: nats)
- Notation: $H(X) = H_p(X) = H(p) = H_X(p) = H(p_X)$

Using the Formula: Example

- Toss a fair coin: $\Omega = \{head, tail\}$
  - p(head) = .5, p(tail) = .5
  - **H(p)** = $- 0.5 \log_2(0.5) + (- 0.5 \log_2(0.5)) =$
    $2 \times ((-0.5) \times (-1)) = 2 \times 0.5 =$ **1**
- Take fair, 32-sided die: p(x) = 1 / 32 for every side x
  - **H(p)** = $-\sum_{i = 1..32} p(x_i) \log_2 p(x_i) = - 32 (p(x_1) \log_2 p(x_1)$
    (since for all *i* $p(x_i) = p(x_1) = 1/32$)
    $= -32 \times ((1/32) \times (-5)) =$ **5** *(now you see why it's called **bits**?)*
- Unfair coin:
  - p(head) = .2 ... **H(p) = .722**; p(head) = .01 ... **H(p) = .081**

Example: Book Availability



The Limits

$\log(x \cdot y) = \log(x) + \log(y)$

- When H(p) = 0?
  - if a result of an experiment is ***known*** ahead of time:
  - necessarily:

$$\mathbf{\exists x \in \Omega; p(x) = 1 \ \& \ \forall y \in \Omega; y \neq x \Rightarrow p(y) = 0}$$

- Upper bound?
  - none in general
  - for $|\Omega| = n$: $H(p) \leq \log_2 n$
    - **nothing can be more uncertain than the uniform distribution**

$\Leftarrow \sum_x -p(x) \cdot \log(p(x)) = \sum_x -\frac{1}{n} \cdot \log(\frac{1}{n}) = -\frac{1}{n} \cdot \sum_x \log(\frac{1}{n}) = -\frac{1}{n} \log_2((\frac{1}{n})^n) = -\frac{1}{n} \cdot n \cdot \log(\frac{1}{n})$
$= -1 \cdot \log(\frac{1}{n}) = \log(n)$

Entropy and Expectation

- Recall:
  - $E(X) = \sum_{x \in X(\Omega)} p_X(x) \times x$
- Then:

$$E(\log_2(1/p_X(x))) = \sum_{x \in X(\Omega)} p_X(x) \log_2(1/p_X(x)) =$$

$$= - \sum_{x \in X(\Omega)} p_X(x) \log_2 p_X(x) =$$

$$= H(p_X) =_{notation} H(p)$$

Perplexity: motivation

- Recall:
  - 2 equiprobable outcomes: H(p) = 1 bit
  - 32 equiprobable outcomes: H(p) = 5 bits
  - 4.3 billion equiprobable outcomes: H(p) ~= 32 bits
- What if the outcomes are not equiprobable?
  - 32 outcomes, 2 equiprobable at .5, rest impossible:
    - **H(p) = 1 bit**
  - Any measure for comparing the entropy (i.e. uncertainty/difficulty of prediction) (also) for random variables with <u>*different number of outcomes*</u>?

size of a vocab with uniform distr, where each word has equal prob.

Perplexity

- Perplexity:
  - $G(p) = 2^{H(p)}$
- ... so we are back at 32 (for 32 eqp. outcomes), 2 for fair coins, etc.
- it is easier to imagine:
  - NLP example: vocabulary size of a vocabulary with uniform distribution, which is equally hard to predict
- the “wilder” (biased) distribution, the better:
  - lower entropy, lower perplexity

Joint Entropy and Conditional Entropy

- Two random variables: X (space Ω),Y (Ψ)
- Joint entropy:
  - no big deal: ((X,Y) considered a single event):

$$H(X,Y) = - \sum_{x \in \Omega} \sum_{y \in \Psi} p(x,y) \log_2 p(x,y)$$

→ just joint inside

- Conditional entropy:

$$H(Y|X) = - \sum_{x \in \Omega} \sum_{y \in \Psi} p(x,y) \log_2 p(y|x)$$

→ unconditional [p(x,y)]; → conditional [p(y|x)]

recall that $H(X) = E(\log_2(1/p_X(x)))$

(weighted "average", and weights are not conditional)

Conditional Entropy (Using the Calculus)

- other definition:

$$H(Y|X) = \sum_{x \in \Omega} p(x)\, H(Y|X=x) =$$

tohle se dá vzít jako obyč entropy, jen pst bude vždy podmíněná

for $H(Y|X=x)$, we can use the single-variable definition ($x \sim$ constant)

$$= \sum_{x \in \Omega} p(x) \Big(- \sum_{y \in \Psi} p(y|x) \log_2 p(y|x) \Big) =$$

$$= - \sum_{x \in \Omega} \sum_{y \in \Psi} p(y|x)\, p(x) \log_2 p(y|x) =$$

$$= - \sum_{x \in \Omega} \sum_{y \in \Psi} p(x,y) \log_2 p(y|x)$$

Properties of Entropy I

- Entropy is non-negative:
  - $H(X) \geq 0$
  - proof: (recall: $H(X) = - \sum_{x \in \Omega} p(x) \log_2 p(x)$)
    - **log(p(x)) is negative or zero for $x \leq 1$,**
    - **p(x) is non-negative; their product p(x)log(p(x) is thus negative;**
    - **sum of negative numbers is negative;**
    - **and $-f$ is positive for negative f**
- Chain rule:
  - $H(X,Y) = H(Y|X) + H(X)$, as well as
  - $H(X,Y) = H(X|Y) + H(Y)$ (since $H(Y,X) = H(X,Y)$)

Když jsou závislé, tak jeden jev říká něco i o tom druhém

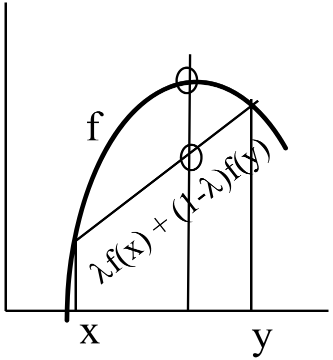
Properties of Entropy II

- Conditional Entropy is better (than unconditional):
  - $H(Y|X) \leq H(Y)$ —> I can only get more information from knowing X
- $H(X,Y) \leq H(X) + H(Y)$ (follows from the previous (in)equalities)
  - **equality iff X,Y independent**
  - **[recall: X,Y independent iff $p(X,Y) = p(X)p(Y)$]**
- H(p) is concave (remember the book availability graph?)
  - concave function f over an interval (a,b):

$$\forall x,y \in (a,b),\ \forall \lambda \in [0,1]:$$

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$$

  - **function f is convex if -f is concave**



- [for proofs and generalizations, see Cover/Thomas]

“Coding” Interpretation of Entropy

- The least (average) number of bits needed to encode a message (string, sequence, series,...) (each element having being a result of a random process with some distribution p): = H(p)
- Remember various compressing algorithms?
  - they do well on data with repeating (= easily predictable = low entropy) patterns
  - their results though have high entropy $\Rightarrow$ compressing compressed data does nothing

Coding: Example

- How many bits do we need for ISO Latin 1?
  - $\Rightarrow$ the trivial answer: 8
- Experience: some chars are more common, some (very) rare:
  - **...so what if we use more bits for the rare, and less bits for the frequent? [be careful: want to decode (easily)!]**
  - **suppose: p('a') = 0.3, p('b') = 0.3, p('c') = 0.3, the rest: p(x)≅ .0004**
  - **code: 'a' ~ 00, 'b' ~ 01, 'c' ~ 10, rest: 11$b_1b_2b_3b_4b_5b_6b_7b_8$**
  - **code acbbécbaac:** `0010010111000011111001000010`

    `a c b b    é    c b a a c`
  - **number of bits used: 28 (vs. 80 using "naive" coding)** 80 = 10 · 8b
- code length ~ 1 / probability; conditional prob OK!

musím ale předem znát, co se bude kanálem posílat...

Entropy of a Language

- Imagine that we produce the next letter using

$$p(l_{n+1}|l_1,...,l_n),$$

l_1 – l_n is what I have seen already, l_{n+1} is what I predict

where $l_1,...,l_n$ is the sequence of ***all*** the letters which had been uttered so far (i.e. ***n*** is really big!); let's call $l_1,...,l_n$ the ***history*** h (h_{n+1}), and all histories H:

- Then compute its entropy:

$$- \sum_{h \in H} \sum_{l \in A} p(l,h) \log_2 p(l|h)$$

- Not very practical, isn't it?

→ tohle je ale hrozně velký

Kullback-Leibler Distance (Relative Entropy)

- Remember:
  - long series of experiments... c_i/T_i oscillates around some number... we can only estimate it... to get a distribution q.
- So we get a distribution q; (sample space Ω, r.v. X)

  the true distribution is, however, p. (same Ω, X)

  $\Rightarrow$ how big error are we making?
- D(p||q) (the Kullback-Leibler distance):

$$D(p\|q) = \sum_{x \in \Omega} p(x) \log_2 (p(x)/q(x)) = E_p \log_2 (p(x)/q(x))$$

observation

unknown

HW2

Comments on Relative Entropy

- Conventions:
  - $0 \log 0 = 0$
  - $p \log (p/0) = \infty$ (for $p > 0$)

might be handy in HW2

- Distance? (less "misleading": Divergence)
  - not quite:
    - **not symmetric: $D(p||q) \neq D(q||p)$**
    - **does not satisfy the triangle inequality**
  - but useful to look at it that way

actually not distance

- $H(p) + D(p||q)$: bits needed for encoding p if q is used

kolik minimálně potřebuju — kolik se ztratí díky jinému kódování

Mutual Information (MI)

in terms of relative entropy

- Random variables X, Y; $p_{X \cap Y}(x,y)$, $p_X(x)$, $p_Y(y)$
- Mutual information (between two random variables X,Y):

KL - Divergence

$$I(X,Y) = D(p(x,y) \,||\, p(x)p(y))$$

- if X independent of Y => ↖ = 0

- I(X,Y) measures how much (our knowledge of) Y contributes (on average) to easing the prediction of X
- or, how p(x,y) deviates from (independent) p(x)p(y)

Mutual Information: the Formula

- Rewrite the definition: [recall: $D(r||s) = \sum_{v \in \Omega} r(v) \log_2 (r(v)/s(v))$;
  substitute $r(v) = p(x,y)$, $s(v) = p(x)p(y)$; $\langle v\rangle \sim \langle x,y\rangle$]

$$I(X,Y) = D(p(x,y) \,||\, p(x)p(y)) =$$
$$= \sum_{x \in \Omega} \sum_{y \in \Psi} p(x,y) \log_2 (p(x,y)/p(x)p(y))$$!

→ the same definition of KL-Divergence

- Measured in bits (what else? :-)

$(p(x,y)/p(x)p(y))$

→ controversely: p(x)·p(y) way higher than p(x,y), they are almost not used together.

→ if p(x,y) is way higher than p(x)·p(y), they are very likely collocations, as they are very much next to each other

From Mutual Information to Entropy

- by how many bits the knowledge of Y ***lowers*** the entropy H(X):

$$I(X,Y) = \sum_{x \in \Omega} \sum_{y \in \Psi} p(x,y) \log_2 (p(x,y)/p(y)p(x)) =$$

...use p(x,y)/p(y) = p(x|y)

$$= \sum_{x \in \Omega} \sum_{y \in \Psi} p(x,y) \log_2 (p(x|y)/p(x)) =$$

...use log(a/b) = log a - log b (a ~ p(x|y), b ~ p(x)), distribute sums

$$= \sum_{x \in \Omega} \sum_{y \in \Psi} p(x,y) \log_2 p(x|y) - \sum_{x \in \Omega} \sum_{y \in \Psi} p(x,y) \log_2 p(x) =$$

...use def. of H(X|Y) (left term), *and* $\sum_{y \in \Psi} p(x,y) = p(x)$ (right term)

$$= - H(X|Y) + (- \sum_{x \in \Omega} p(x) \log_2 p(x)) =$$

...use def. of H(X) (right term), *swap terms*

$$= H(X) - H(X|Y) \qquad \text{...by symmetry,} = H(Y) - H(Y|X)$$

Properties of MI vs. Entropy

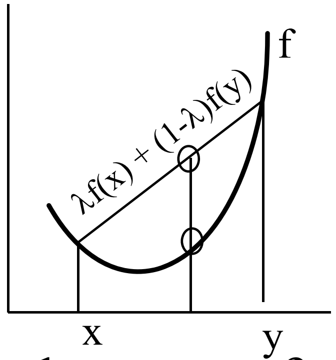
- $I(X,Y) = H(X) - H(X|Y)$ = number of bits the knowledge of Y lowers the entropy of X

  $= H(Y) - H(Y|X)$ (prev. foil, symmetry)

Recall: $H(X,Y) = H(X|Y) + H(Y) \Rightarrow -H(X|Y) = H(Y) - H(X,Y) \Rightarrow$

- $I(X,Y) = H(X) + H(Y) - H(X,Y)$
- $I(X,X) = H(X)$ (since $H(X|X) = 0$)
- $I(X,Y) = I(Y,X)$ (just for completeness)
- $I(X,Y) \geq 0$ ... let's prove that now (as promised).

Jensen's Inequality



- Recall: <u>f</u> is convex on interval (a,b) iff

  $\forall x,y \in (a,b), \forall \lambda \in [0,1]$:

  $$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

- J.I.: for distribution p(x), r.v. X on Ω, and convex f,

  $$f\left(\sum_{x\in\Omega} p(x)\, x\right) \leq \sum_{x\in\Omega} p(x)\, f(x)$$

- <u>Proof</u> (idea): by induction on the number of basic outcomes;
- start with $|\Omega| = 2$ by: → this is rewritten definition of Jensen's inequality...
  - **$p(x_1)f(x_1) + p(x_2)f(x_2) \geq f(p(x_1)x_1 + p(x_2)x_2)$ (⇐ def. of convexity)**
  - **for the induction step ($|\Omega| = k \rightarrow k+1$), just use the induction hypothesis and def. of convexity (again).**

Information Inequality

$$D(p||q) \geq 0 \;!$$

- Proof:

$$0 = -\log 1 = -\log \sum_{x\in\Omega} q(x) = -\log \sum_{x\in\Omega} (q(x)/p(x))p(x) \leq$$

...apply Jensen's inequality here (- log is convex)... → just rewritten...

$$\leq \sum_{x\in\Omega} p(x)\,(-\log(q(x)/p(x))) = \sum_{x\in\Omega} p(x) \log(p(x)/q(x)) =$$

$$= D(p||q)$$

Other (In)Equalities and Facts

- Log sum inequality: for $r_i, s_i \geq 0$

  $$\sum_{i=1..n} (r_i \log(r_i/s_i)) \overset{\geq}{\not\leq} (\sum_{i=1..n} r_i) \log(\sum_{i=1..n} r_i / \sum_{i=1..n} s_i))$$

- $D(p||q)$ is convex [in p,q] ($\Leftarrow$ log sum inequality)
- $H(p_X) \leq \log_2|\Omega|$, where Ω is the sample space of p_X

  Proof: uniform $u(x)$, same sample space Ω: $\sum p(x) \log u(x) = -\log_2|\Omega|$;

  $$\log_2|\Omega| - H(X) = -\sum p(x) \log u(x) + \sum p(x) \log p(x) = D(p||u) \geq 0$$

- $H(p)$ is concave [in p]:

  Proof: from $H(X) = \log_2|\Omega| - D(p||u)$, $D(p||u)$ convex $\Rightarrow H(x)$ concave

Cross-Entropy

- Typical case: we've got series of observations

  $T = \{t_1, t_2, t_3, t_4, ..., t_n\}$ (numbers, words, ...; $t_i \in \Omega$);

  estimate (simple):

  $\forall y \in \Omega$: $\tilde{p}(y) = c(y) / |T|$, def. $c(y) = |\{t \in T; t = y\}|$
- ...but the true p is unknown; every sample is too small!
- Natural question: how well do we do using $\tilde{p}$ [instead of p]?
- Idea: simulate actual p by using a different T'

  (or rather: by using different observation we simulate the insufficiency of T vs. some other data ("random" difference))

Cross Entropy: The Formula

- $H_{p'}(\tilde{p}) = H(p') + D(p' \| \tilde{p})$

→ entropy of what we expect to be real

→ entropy of what we need to get close to real

$$H_{p'}(\tilde{p}) = - \sum_{x \in \Omega} p'(x) \log_2 \tilde{p}(x) \quad !$$

Same as normal entropy, only taking prob. from expected distr. and counting with observed one

- p' is certainly not the true p, but we can consider it the "real world" distribution against which we test $\tilde{p}$
- note on notation (confusing...): $p/p' \leftrightarrow \tilde{p}$, also $H_{T'}(p)$
- (Cross)Perplexity: $G_{p'}(p) = G_{T'}(p) = 2^{H_{p'}(\tilde{p})}$

Conditional Cross Entropy

- So far: "unconditional" distribution(s) p(x), p'(x)...
- In practice: virtually always conditioning on context
- Interested in: sample space Ψ, r.v. Y, $y \in \Psi$;

  context: sample space Ω, r.v. X, $x \in \Omega$;:

  "our" distribution p(y|x), test against p'(y,x),

  which is taken from some independent data:

$$H_{p'}(p) = - \sum_{y \in \Psi, x \in \Omega} p'(y,x) \log_2 p(y|x)$$

"average of the entropy of guessing next element" <-> again this is regular joint prob.

Sample Space vs. Data

- In practice, it is often inconvenient to sum over the sample space(s) Ψ, Ω (especially for cross entropy!)
- Use the following formula:

$$H_{p'}(p) = - \sum_{y \in \Psi, x \in \Omega} p'(y,x) \log_2 p(y|x) = - 1/|T'| \sum_{i = 1..|T'|} \log_2 p(y_i|x_i) \quad !$$

if this space is small enough, you can iterate over all, but usually it would take ages

iteration over all data T', go over whole sequence T'

|T'| is the sample size

the "p" dist is given (by the model např.)

- This is in fact the normalized log probability of the "test" data:

$$H_{p'}(p) = - 1/|T'| \log_2 \prod_{i = 1..|T'|} p(y_i|x_i)$$

tohle chceme ideálně použít

b r e a k
1 6 6 2 6 = 21/5

Computation Example

- Ω = {a,b,..,z}, prob. distribution (assumed/estimated from data):
  p(a) = .25, p(b) = .5, p(α) = 1/64 for α ∈{c..r}, = 0 for the rest: s,t,u,v,w,x,y,z
  given by language model
- Data (test): barb   p'(a) = p'(r) = .25, p'(b) = .5
  → this is our observation (test data)
- Sum over Ω:

α	a	b	c	d	e	f	g	...	p	q	r	s	t	...	z	
$-p'(\alpha)\log_2 p(\alpha)$	.5+	.5+	0+	0+	0+	0+	0+	0+	0+	0+	1.5+	0+	0+	0+	0+0	= **2.5**

- Sum over data:

i / s_i	1/b		2/a		3/r		4/b			1/|T'|
$-\log_2 p(s_i)$	1	+	2	+	6	+	1	= 10	(1/4) × 10 = **2.5**	

?

Cross Entropy: Some Observations

- H(p) ?? <, =, > ?? $H_{p'}(p)$: ALL!
- Previous example:

  [p(a) = .25, p(b) = .5, p(α) = 1/64 for α ∈{c..r}, = 0 for the rest: s,t,u,v,w,x,y,z]

  H(p) = 2.5 bits = H(p') (barb)

- Other data: probable: **(1/8)(6+6+6+1+2+1+6+6)= 4.25**

  H(p) < 4.25 bits = H(p') (probable)

- And finally: abba: **(1/4)(2+1+1+2)= 1.5**

  H(p) > 1.5 bits = H(p') (abba)

- But what about: baby $-p'(\text{'y'})\log_2 p(\text{'y'}) = -.25\log_2 0 = \infty$ (??)

↳ for this reason it makes sense to assign at least a very small prob to all values.

Cross Entropy: Usage

- Comparing data?? Think of it that all data are taken from SOME distributions
  - *NO!* (we believe that we test on ***real*** data!)
- Rather: comparing distributions (***vs.*** real data)
- Have (got) 2 distributions: p and q (on some Ω, X)
  - which is better? → is modeling the reality better
  - better: has lower cross-entropy (perplexity) on real data S
- "Real" data: S
- $H_S(p) = - 1/|S| \sum_{i = 1..|S|} \log_2 p(y_i|x_i)$?? $H_S(q) = - 1/|S| \sum_{i = 1..|S|} \log_2 q(y_i|x_i)$

Comparing Distributions

Test data S: probable

- p(.) from prev. example: $H_S(p) = 4.25$

  $p(a) = .25$, $p(b) = .5$, $p(\alpha) = 1/64$ for $\alpha \in \{c..r\}$, $= 0$ for the rest: s,t,u,v,w,x,y,z

- q(.|.) (conditional; defined by a table): „L" is always followed by „E"

q(.|.)→ ↓	a	b	e	l	o	p	r	other
a	0	.5	0	0	0	.125	0	0
b	1	0	0	0	1	.125	0	0
e	0	0	0	1	0	.125	0	0
l	0	.5	0	0	0	.125	0	0
o	0	0	0	0	0	.125	1	0
p	0	0	0	0	0	.125	0	1
r	0	0	0	0	0	.125	0	0
other	0	0	1	0	0	.125	0	0

ex.: $q(o|r) = 1$

$q(r|p) = .125$

0,625 < 4,25

(1/8) (log(p|oth.)+log(r|p)+log(o|r)+log(b|o)+log(a|b)+log(b|a)+log(l|b)+log(e|l))

(1/8) (0 + 3 + 0 + 0 + 1 + 0 + 1 + 0)

„over the data" way

$H_S(q) = .625$

=> this tells me that this model is wayyyy closer to the real S.

Language Modeling
(and the Noisy Channel)

SpeechRecognition --> modeling goes from text to speech

The Noisy Channel

- Prototypical case:

Input		Output (noisy)
0,1,1,1,0,1,0,1,... →	The channel (adds noise)	→ 0,1,1,0,0,1,1,0,...

- Model: probability of error (noise):
- Example: p(0|1) = .3 p(1|1) = .7 p(1|0) = .4 p(0|0) = .6

-> pravděp. že p(x|y)
y se přepíše na x

- The Task:

known: the noisy output; want to know: the input (***decoding***)

Noisy Channel Applications

- OCR
  - straightforward: text $\rightarrow$ print (adds noise), scan $\rightarrow$ image
- Handwriting recognition
  - text $\rightarrow$ neurons, muscles (“noise”), scan/digitize $\rightarrow$ image
- Speech recognition (dictation, commands, etc.)
  - text $\rightarrow$ conversion to acoustic signal (“noise”) $\rightarrow$ acoustic waves
- Machine Translation
  - text in target language $\rightarrow$ translation (“noise”) $\rightarrow$ source language
- Also: Part of Speech Tagging
  - sequence of tags $\rightarrow$ selection of word forms $\rightarrow$ text

Noisy Channel: The Golden Rule of ...

OCR, ASR, HR, MT, ...

- Recall:

  $p(A|B) = p(B|A)\ p(A) / p(B)$ (Bayes formula)

  $A_{best} = \mathrm{argmax}_A\ p(B|A)\ p(A)$ (The Golden Rule)

-> I know B... (it is the output)

this can be ignored, since it is still the same

- $p(B|A)$: the acoustic/image/translation/lexical model
  - application-specific name
  - will explore later
- $p(A)$: ***the language model***

The Perfect Language Model

- Sequence of word forms [forget about tagging for the moment]
- Notation: $A \sim W = (w_1, w_2, w_3, \ldots, w_d)$ —> think of it as "d" words
- The big (modeling) question:

$$p(W) = ?$$

- Well, we know (Bayes/chain rule →):

$$p(W) = p(w_1, w_2, w_3, \ldots, w_d) =$$

$$= p(w_1) \times p(w_2|w_1) \times p(w_3|w_1,w_2) \times \ldots \times p(w_d|w_1,w_2,\ldots,w_{d-1})$$

- Not practical (even short W → too many parameters)

this is still too long

Markov Chain

- Unlimited memory (cf. previous foil):
  - for w_i, we know all its predecessors $w_1, w_2, w_3, \ldots, w_{i-1}$
- Limited memory:
  - we disregard "too old" predecessors → this is our assumption (for approximation) Markov Assumption
  - remember only *k* previous words: $w_{i-k}, w_{i-k+1}, \ldots, w_{i-1}$
  - called "k^{th} order Markov approximation"
- + stationary character (no change over time):

$$p(W) \cong \Pi_{i=1..d} p(w_i | w_{i-k}, w_{i-k+1}, \ldots, w_{i-1}),\ d = |W|$$

→ this k-tuple we call state

n-gram Language Models

- $(n-1)^{th}$ order Markov approximation → n-gram LM:

prediction — history (n-1 words)

$$p(W) =_{df} \prod_{i=1..d} p(w_i|w_{i-n+1},w_{i-n+2},\dots,w_{i-1})$$

!

- In particular (assume vocabulary $|V| = 60k$):
  - **0-gram LM: uniform model, $p(w) = 1/|V|$, 1 parameter**
  - **1-gram LM: unigram model, $p(w)$, 6×10^4 parameters**
  - **2-gram LM: bigram model, $p(w_i|w_{i-1})$ 3.6×10^9 parameters**
  - **3-gram LM: trigram model, $p(w_i|w_{i-2},w_{i-1})$ 2.16×10^{14} parameters**

we don't need to count them

↳ actually 3-grams are still feasable, because a lot of combinations have prob. = 0 since they are impossible...

- if the n is too big, then entropy would be very low, but also the prob. of seeing a similar history of n-1 words. Therefor such a model would be bad at generalizing

LM: Observations

- How large *n*?
  - nothing is enough (theoretically)
  - but anyway: as much as possible (→ close to "perfect" model)
  - empirically: ***3***
    - **parameter estimation? (reliability, data availability, storage space, ...)**
    - **4 is too much: |V|=60k → 1.296×10^{19} parameters**
    - **but: 6-7 would be (almost) ideal (having enough data): in fact, one can recover the original text ssequence from 7-grams!**
- Reliability ~ (1 / Detail) (→ need compromise)
- For now, keep word forms (no "linguistic" processing)

The Length Issue

- $\forall n; \Sigma_{w \in \Omega^n} p(w) = 1 \Rightarrow \Sigma_{n=1..\infty} \Sigma_{w \in \Omega^n} p(w) >> 1 \; (\rightarrow \infty)$
- We want to model all sequences of words
  - for "fixed" length tasks: no problem - n fixed, sum is 1
    - **tagging, OCR/handwriting (if words identified ahead of time)**
  - for "variable" length tasks: have to account for
    - **discount shorter sentences**
- General model: for each sequence of words of length n,

  define $p'(w) = \lambda_n p(w)$ such that $\Sigma_{n=1..\infty} \lambda_n = 1 \Rightarrow$

  $$\Sigma_{n=1..\infty} \Sigma_{w \in \Omega^n} p'(w) = 1$$

  e.g., estimate λ_n from data; or use normal or other distribution

this must be tuned on the text

Parameter Estimation

- Parameter: numerical value needed to compute p(w|h)
- From data (how else?)
- Data preparation:
  - **get rid of formatting etc. ("text cleaning")**

    jako třeba html věci...
  - **define words (separate but include punctuation, call it "word")**
  - **define sentence boundaries (insert "words" <s> and </s>)**
  - **letter case: keep, discard, or be smart:**
    - name recognition
    - number type identification

    [these are huge problems per se!]
  - **numbers: keep, replace by <num>, or be smart (form ~ pronunciation)**

Maximum Likelihood Estimate

- MLE: Relative Frequency...
  - ...best predicts the data at hand (the “training data”)
- Trigrams from Training Data T:
  - count sequences of three words in T: $c_3(w_{i-2},w_{i-1},w_i)$

    [NB: notation: just saying that the three words follow each other]
  - count sequences of two words in T: $c_2(w_{i-1},w_i)$:
    - **either use $c_2(y,z) = \Sigma_w c_3(y,z,w)$**
    - **or count differently at the beginning (& end) of data!**

also should be used in our homework

→ you always do the counts for the most detailed ones.

→ because of word endings / beginnings

$$\mathbf{p(w_i|w_{i-2},w_{i-1}) =_{est.} c_3(w_{i-2},w_{i-1},w_i) / c_2(w_{i-2},w_{i-1})}$$!

otherwise it doesn't have to sum to 1.

Character Language Model

- Use individual characters instead of words:

$$p(W) =_{df} \prod_{i=1..d} p(c_i | c_{i-n+1}, c_{i-n+2}, \ldots, c_{i-1})$$

- Same formulas etc.
- Might consider 4-grams, 5-grams or even more
- Good only for language comparison
- Transform cross-entropy between letter- and word-based models:

$$H_S(p_c) = H_S(p_w) \text{ / avg. \# of characters/word in S}$$

$$H(p) = \sum_{x} p(x) \cdot \log(p(x))$$

LM: an Example

- Training data:

  <s> <s> He can buy the can of soda.

  - Unigram: p_1(He) = p_1(buy) = p_1(the) = p_1(of) = p_1(soda) = p_1(.) = .125
    p_1(can) = .25
  - Bigram: p_2(He|<s>) = 1, p_2(can|He) = 1, p_2(buy|can) = .5,
    p_2(of|can) = .5, p_2(the|buy) = 1,...
  - Trigram: p_3(He|<s>,<s>) = 1, p_3(can|<s>,He) = 1,
    p_3(buy|He,can) = 1, p_3(of|the,can) = 1, ..., p_3(.|of,soda) = 1.
  - Entropy: $H(p_1) = 2.75$, $H(p_2) = .25$, $H(p_3) = 0$ ← Great?!

LM: an Example (The Problem)

- Cross-entropy:
- S = <s> <s> It was the greatest buy of all.
- Even $H_S(p_1)$ fails (= $H_S(p_2) = H_S(p_3) = \infty$), because:
  - all unigrams but p_1(the), p_1(buy), p_1(of) and p_1(.) are 0.
  - all bigram probabilities are 0.
  - all trigram probabilities are 0.
- We want: to make all (theoretically possible*) probabilities non-zero.

*in fact, <u>all</u>: remember our graph from day 1?

LM Smoothing
(And the EM Algorithm)

You would discard the prediction completely ~ having any +∞ in the sum

The Zero Problem

- “Raw” n-gram language model estimate:
  - necessarily, some zeros
    - **!many: trigram model → 2.16×10^{14} parameters, data ~ 10^{9} words**
  - which are true 0?
    - **optimal situation: even the least frequent trigram would be seen several times, in order to distinguish it’s probability vs. other trigrams**
    - **optimal situation cannot happen, unfortunately (open question: how many data would we need?)**
  - → we don’t know

    tohle je OK

    tady nahradíme uniform distr.
  - we must eliminate the zeros
- Two kinds of zeros: $p(w|h) = 0$, or even $p(h) = 0$!

Why do we need Nonzero Probs?

- To avoid infinite Cross Entropy:
  - happens when an event is found in test data which has not been seen in training data

    $H(p) = \infty$: prevents comparing data with > 0 "errors"
- To make the system more robust
  - low count estimates:
    - **they typically happen for "detailed" but relatively rare appearances**
  - high count estimates: reliable but less "detailed"

Eliminating the Zero Probabilities: Smoothing

- Get new p'(w) (same Ω): almost p(w) but no zeros
- Discount w for (some) p(w) > 0: new p'(w) < p(w)

$$\Sigma_{w \in \text{discounted}} (p(w) - p'(w)) = D$$

- Distribute D to all w; p(w) = 0: new p'(w) > p(w)
  - possibly also to other w with low p(w)
- For some w (possibly): p'(w) = p(w)
- Make sure $\Sigma_{w \in \Omega} p'(w) = 1$
- There are many ways of ***smoothing***

can't add just to 0-occurrence sequences, because they will become same as 1-occurrence seq.

Smoothing by Adding 1

- Simplest but not really usable:
  - Predicting words w from a vocabulary V, training data T:

    p'(w|h) = (c(h,w) + 1) / (c(h) + |V|)   Not OK

    - **for non-conditional distributions: p'(w) = (c(w) + 1) / (|T| + |V|)**
  - Problem if |V| > c(h) (as is often the case; even >> c(h)!)
- Example: Training data: <s> what is it what is small ? |T| = 8
  - **V = { what, is, it, small, ?, <s>, flying, birds, are, a, bird, . }, |V| = 12**
  - **p(it)=.125, p(what)=.25, p(.)=0 p(what is it?) = $.25^2 \times .125^2 \cong$ .001**

    p(it is flying.) = $.125 \times .25 \times 0^2 = 0$
  - **p'(it) =.1, p'(what) =.15, p'(.)=.05 p'(what is it?) = $.15^2 \times .1^2 \cong$ .0002**

    p'(it is flying.) = $.1 \times .15 \times .05^2 \cong .00004$

→ mathematically works the same, performs much better

Adding *less than* 1

we can find the best lambda by getting the lowest CE

→ this would be 0 for training data in cross-validation

- Equally simple:
  - Predicting words w from a vocabulary V, training data T:

    $p'(w|h) = (c(h,w) + \lambda) / (c(h) + \lambda|V|), \lambda < 1$

    - **for non-conditional distributions: $p'(w) = (c(w) + \lambda) / (|T| + \lambda|V|)$**
- Example: Training data: <s> what is it what is small ? $|T| = 8$
    - **V = { what, is, it, small, ?, <s>, flying, birds, are, a, bird, . }, |V| = 12**
    - **p(it)=.125, p(what)=.25, p(.)=0 p(what is it?) = $.25^2 \times .125^2 \cong$ .001**

      p(it is flying.) = $.125 \times .25 \times 0^2 = 0$
    - **Use $\lambda = .1$:**
    - **p'(it)$\cong$ .12, p'(what)$\cong$ .23, p'(.)$\cong$ .01 p'(what is it?) = $.23^2 \times .12^2 \cong$ .0007**

      p'(it is flying.) = $.12 \times .23 \times .01^2 \cong$ .000003

Good - Turing

Here you dont need to do any optimalization to obtain any lambda...

- Suitable for estimation from large data
  - similar idea: discount/boost the relative frequency estimate:

    $\mathbf{p_r(w) = (c(w) + 1) \times N(c(w) + 1) / (|T| \times N(c(w)))}$,

    where N(c) is the count of words with count c (count-of-counts)

    specifically, for c(w) = 0 (unseen words), $\mathbf{p_r(w) = N(1) / (|T| \times N(0))}$
  - good for small counts (< 5-10, where N(c) is high)
  - variants (see MS)
  - normalization! (so that we have $\Sigma_w \, p'(w) = 1$)

Good-Turing: An Example

- Example: remember: $p_r(w) = (c(w) + 1) \times N(c(w) + 1) / (|T| \times N(c(w)))$

  Training data: <s> what is it what is small ? $|T| = 8$

  - **V = { what, is, it, small, ?, <s>, flying, birds, are, a, bird, . }, |V| = 12**

    p(it)=.125, p(what)=.25, p(.)=0 p(what is it?) = $.25^2 \times .125^2 \cong$ .001

    p(it is flying.) = $.125 \times .25 \times 0^2 =$ 0

  - **Raw reestimation (N(0) = 6, N(1) = 4, N(2) = 2, N(i) = 0 for i > 2):**

    $p_r(it) = (1+1) \times N(1+1)/(8 \times N(1)) = 2 \times 2/(8 \times 4) = .125$

    $p_r(what) = (2+1) \times N(2+1)/(8 \times N(2)) = 3 \times 0/(8 \times 2) = 0$: keep orig. p(what)

    $p_r(.) = (0+1) \times N(0+1)/(8 \times N(0)) = 1 \times 4/(8 \times 6) \cong .083$

  - **Normalize (divide by $1.5 = \Sigma_{w \in |V|} p_r(w)$) and compute:**

    p'(it)≅ .08, p'(what)≅ .17, p'(.)≅ .06 p'(what is it?) = $.17^2 \times .08^2 \cong$ .0002

    p'(it is flying.) = $.08 \times .17 \times .06^2 \cong$ .00004

Smoothing by Combination: Linear Interpolation

- Combine what?
  - **distributions of various level of detail vs. reliability**
- n-gram models:
  - **use (n-1)gram, (n-2)gram, ..., uniform**

    → reliability

    ← detail

  –> if we combine them, we get the best results
- Simplest possible combination:
  – sum of probabilities, normalize:
    - **p(0|0) = .8, p(1|0) = .2, p(0|1) = 1, p(1|1) = 0, p(0) = .4, p(1) = .6:**
    - **p’(0|0) = .6, p’(1|0) = .4, p’(0|1) = .7, p’(1|1) = .3**

typically you won't count weights, but say why and how it works

Typical n-gram LM Smoothing

- Weight in less detailed distributions using $\lambda=(\lambda_0,\lambda_1,\lambda_2,\lambda_3)$:

  $$p'_\lambda(w_i|\, w_{i-2}, w_{i-1}) = \lambda_3\, p_3(w_i|\, w_{i-2}, w_{i-1}) + \lambda_2\, p_2(w_i|\, w_{i-1}) + \lambda_1\, p_1(w_i) + \lambda_0 / |V|$$

  originally uniform 1/|V|

- Normalize:

  $\lambda_i > 0$, $\Sigma_{i=0..n} \lambda_i = 1$ is sufficient ($\lambda_0 = 1 - \Sigma_{i=1..n} \lambda_i$) (n=3)

- Estimation using MLE:
  - fix the p_3, p_2, p_1 and $|V|$ parameters as estimated from the training data
  - then find such $\{\lambda_i\}$ which minimizes the cross entropy (maximizes probability of data): $-(1/|D|)\Sigma_{i=1..|D|}\log_2(p'_\lambda(w_i|h_i))$

until happy

Held-out Data

- What data to use?
  - try the training data T: but we will always get $\lambda_3 = 1$
    - **why? (let p_{iT} be an i-gram distribution estimated using r.f. from T)**
    - **minimizing $H_T(p'_\lambda)$ over a vector λ, $p'_\lambda = \lambda_3 p_{3T} + \lambda_2 p_{2T} + \lambda_1 p_{1T} + \lambda_0/|V|$**
      - remember: $H_T(p'_\lambda) = H(p_{3T}) + D(p_{3T}||p'_\lambda)$; p_{3T} fixed $\rightarrow$ $H(p_{3T})$ fixed, and it is the best
      - which p'_λ minimizes $H_T(p'_\lambda)$? ... a p'_λ for which $D(p_{3T}|| p'_\lambda)=0$
      - ...and that's p_{3T} (because $D(p||p) = 0$, as we know).
      - ...and certainly $p'_\lambda = p_{3T}$ if $\lambda_3 = 1$ (maybe in some other cases, too).
      - $(p'_\lambda = 1 \times p_{3T} + 0 \times p_{2T} + 0 \times p_{1T} + 0/|V|)$
  - thus: do <u>not use the training data for estimation of λ!</u>
    - ***<u>hold out</u>* part of the training data (*heldout* <u>H</u>); remaining data: the true/raw *training* data, <u>T</u>**
    - **the *test* data <u>S</u> (e.g., for comparison purposes): still must be some different data!**

The Formulas

- Repeat: minimizing $-(1/|H|)\Sigma_{i=1..|H|}\log_2(p'_\lambda(w_i|h_i))$ over λ

$$p'_\lambda(w_i|\, h_i) = p'_\lambda(w_i|\, w_{i-2}, w_{i-1}) = \lambda_3\, p_3(w_i|\, w_{i-2}, w_{i-1}) + \lambda_2\, p_2(w_i|\, w_{i-1}) + \lambda_1\, p_1(w_i) + \lambda_0 /|V|$$

- "Expected Counts (of lambdas)": $j = 0..3$

$$c(\lambda_j) = \Sigma_{i=1..|H|} (\lambda_j p_j(w_i|h_i) / p'_\lambda(w_i|h_i))$$

→ how many times in the data, was it better to look at j-gram instead of the (latest) combined probability.

- "Next λ": $j = 0..3$

$$\lambda_{j,next} = c(\lambda_j) / \Sigma_{k=0..3} (c(\lambda_k))$$

→ sums to 1

The (Smoothing) EM Algorithm

1. Start with some λ, such that $\lambda_j > 0$ for all $j \in 0..3$.
2. Compute "Expected Counts" for each λ_j.
3. Compute new set of λ_j, using the "Next λ" formula.
4. Start over at step 2, unless a termination condition is met.

- Termination condition: convergence of λ.
  - Simply set an ε, and finish if $|\lambda_j - \lambda_{j,next}| < \varepsilon$ for each j (step 3).
- Guaranteed to converge:

  follows from Jensen's inequality, plus a technical proof.

bucket 1 := highest frequency bucket, ... bucket m := 1-frequency bucket

Problem with buckets: if the data is very small, the bucketing can almost just split the text into 1000 parts and their prob. will correspond to some trivial distr.

Remark on Linear Interpolation Smoothing

→ I got only one lambda per all e.g. 3-grams, even though some have much higher frequency

- "Bucketed" smoothing:
  - use several vectors of λ instead of one, based on (the frequency of) history: λ(h)
    - **e.g. for h = (micrograms,per) we will have**

      λ(h) = (.999,.0009,.00009,.00001)

      (because "cubic" is the only word to follow...)
  - actually: not a separate set for each history, but rather a set for "similar" histories ("bucket"):

    $\lambda(b(h))$, where b: $V^2 \rightarrow N$ (in the case of trigrams)

    b classifies histories according to their reliability (~ frequency)

Bucketed Smoothing: The Algorithm

- First, determine the bucketing function b (use heldout!):
  - decide in advance you want e.g. 1000 buckets
  - compute the total frequency of histories in 1 bucket ($f_{max}(b)$)
  - gradually fill your buckets from the most frequent bigrams so that the sum of frequencies does not exceed $f_{max}(b)$ (you might end up with slightly more than 1000 buckets)
- Divide your heldout data according to buckets
- Apply the previous algorithm to each bucket and its data

Simple Example

- Raw distribution (unigram only; smooth with uniform):

  $p(a) = .25$, $p(b) = .5$, $p(\alpha) = 1/64$ for $\alpha \in \{c..r\}$, $= 0$ for the rest: s,t,u,v,w,x,y,z
- Heldout data: <u>baby</u>; use one set of λ (λ_1: unigram, λ_0: uniform)
- Start with $\lambda_1 = .5$; $p'_\lambda(b) = .5 \text{ x } .5 + .5 / 26 = .27$

  $p'_\lambda(a) = .5 \text{ x } .25 + .5 / 26 = .14$

  $p'_\lambda(y) = .5 \text{ x } 0 + .5 / 26 = .02$

  $c(\lambda_1) = .5\text{x}.5/.27 + .5\text{x}.25/.14 + .5\text{x}.5/.27 + .5\text{x}0/.02 = 2.72$

  $c(\lambda_0) = .5\text{x}.04/.27 + .5\text{x}.04/.14 + .5\text{x}.04/.27 + .5\text{x}.04/.02 = 1.28$

  Normalize: $\lambda_{1,next} = .68$, $\lambda_{0,next} = .32$.

  Repeat from step 2 (recompute p'_λ first for efficient computation, then $c(\lambda_i)$, ...)

  Finish when new lambdas almost equal to the old ones (say, < 0.01 difference).

Some More Technical Hints

- Set V = {all words from training data}.
  - **You may also consider V = T ∪ H, but it does not make the coding in any way simpler (in fact, harder).**
  - **But: you must *never* use the test data for you vocabulary!**
- Prepend two "words" in front of all data:
  - **avoids beginning-of-data problems**
  - **call these index -1 and 0: then the formulas hold exactly**
- When $c_n(w,h) = 0$:
  - **Assign 0 probability to $p_n(w|h)$ where $c_{n-1}(h) > 0$, but a uniform probability (1/|V|) to those $p_n(w|h)$ where $c_{n-1}(h) = 0$ [this must be done both when working on the heldout data during EM, as well as when computing cross-entropy on the test data!]**

Words and the Company They Keep

Motivation

- Environment:
  - mostly “not a full analysis (sentence/text parsing)”
- Tasks where “words & company” are important:
  - word sense disambiguation (MT, IR, TD, IE)
  - lexical entries: subdivision & definitions (lexicography)
  - language modeling (generalization, [kind of] smoothing)
  - word/phrase/term translation (MT, Multilingual IR)
  - NL generation (“natural” phrases) (Generation, MT)
  - parsing (lexically-based selectional preferences)

Collocations

- Collocation
  - Firth: “word is characterized by the company it keeps”; collocations of a given word are statements of the habitual or customary places of that word.
  - non-compositionality of meaning
    - **cannot be derived directly from its parts (<u>heavy</u> rain)**
  - non-substitutability in context
    - **for parts (<u>red</u> light)**
  - non-modifiability (& non-transformability)
    - **kick the <u>yellow</u> bucket; take ex~~ceptions~~ to**

Association and Co-occurence; Terms

- Does not fall under “collocation”, but:
- Interesting just because it does often [rarely] appear together or in the same (or similar) context:
    - **(doctors, nurses)**
    - **(hardware,software)**
    - **(gas, fuel)**
    - **(hammer, nail)**
    - **(communism, free speech)**
- Terms:
    – need not be > 1 word (notebook, washer)

Collocations of Special Interest

- Idioms: really fixed phrases
    - **kick the bucket, birds-of-a-feather, run for office**
- Proper names: difficult to recognize even with lists
    - **Tuesday (person's name), May, Winston Churchill, IBM, Inc.**
- Numerical expressions
  - containing "ordinary" words
    - **Monday Oct 04 1999, two thousand seven hundred fifty**
- Phrasal verbs
  - Separable parts:
    - **look up, take off**

Further Notions

- Synonymy: different form/word, same meaning:
  - **notebook / laptop**
- Antonymy: opposite meaning:
  - **new/old, black/white, start/stop**
- Homonymy: same form/word, different meaning:
  - **“true” (random, unrelated): can (aux. verb / can of Coke)**
  - **related: polysemy; notebook, shift, grade, ...**
- Other:
  - **Hyperonymy/Hyponymy: general vs. special: vehicle/car**
  - **Meronymy/Holonymy: whole vs. part: body/leg**

How to Find Collocations?

- Frequency
  - plain
  - filtered → musím vyfiltrovat „to be" etc.
- Hypothesis testing
  - *t* test
  - χ^2 test
- Pointwise ("poor man's") Mutual Information
- (Average) Mutual Information

Frequency

- Simple
  - Count n-grams; high frequency n-grams are candidates:
    - **mostly function words**
    - **frequent names**
- Filtered
  - Stop list: words/forms which (we think) cannot be a part of a collocation
    - **a, the, and, or, but, not, ...** → these are usually quite small
  - Part of Speech (possible collocation patterns)
    - **A+N, N+N, N+of+N, ...**

Hypothesis Testing

- Hypothesis
  - something we test (against)
- Most often:
  - compare possibly interesting thing vs. “random” chance
  - “Null hypothesis”:
    - **something occurs by chance (that’s what we suppose).**
    - **Assuming this, prove that the probabilty of the “real world” is then too low (typically < 0.05, also 0.005, 0.001)... therefore reject the null hypothesis (thus confirming “interesting” things are happening!)**
    - **Otherwise, it’s possibile there is nothing interesting.**

t test (Student's *t* test)

- Significance of difference
  - compute "magic" number against normal distribution (mean μ)
  - using real-world data: (x' real data mean, s^2 variance, N size):
    - **$t = (x' - \mu) / \sqrt{s^2 / N}$**
  - find in tables (see MS, p. 609):
    - **d.f. = degrees of freedom (parameters which are not determined by other parameters)**
    - **percentile level p = 0.05 (or better)**
  - the bigger t:
    - **the better chances that there is the interesting feature we hope for (i.e. we can reject the null hypothesis)**
    - **t: at least the value from the table(s)**

t test on words

- null hypothesis: independence
  - **mean μ: $p(w_1)\ p(w_2)$**
- data estimates:
  - **x' = MLE of joint probability from data**
  - **s^2 is p(1-p), i.e. almost p for small p; N is the data size**
- Example: (d.f. ~ sample size)
  - **'general term' (homework corpus): c(general) = 108, c(term) = 40**
  - **c(general,term) = 2; expected p(general)p(term) = 8.8E-8**
  - **t = (9.0E-6 - 8.8E-8) / (9.0E-6 / 221097)$^{1/2}$ = 1.40 (not > 2.576) thus 'general term' is <u>not</u> a collocation with confidence 0.005**
  - **'true species': (84/1779/9): t = 2.774 > 2.576 !!**

Pearson's Chi-square test

- χ^2 test (general formula): $\Sigma_{i,j} (O_{ij}-E_{ij})^2 / E_{ij}$
  - where O_{ij}/E_{ij} is the observed/expected count of events i, j
- for two-outcomes-only events:

$w_{right} \setminus w_{left}$	= true	≠ true
= species	9	1,770
≠ species	75	219,243

$\chi^2 =$ 221097(219243x9-75x1770)^2/(1779x84x221013x219318) = 103.39 > 7.88

(at .005 thus we can reject the independence assumption)

Pointwise Mutual Information

- This is ***NOT*** the MI as defined in Information Theory
  - (IT: average of the following; not of ***values***)
- ...but might be useful:

  $I'(a,b) = \log_2 (p(a,b) / p(a)p(b)) = \log_2 (p(a|b) / p(a))$
- Example (same):

  $\mathbf{I'(true,species) = \log_2 (4.1e\text{-}5 / 3.8e\text{-}4 \times 8.0e\text{-}3) = 3.74}$

  $\mathbf{I'(general,term) = \log_2 (9.0e\text{-}6 / 1.8e\text{-}4 \times 4.9e\text{-}4) = 6.68}$
  - **measured in bits but it is difficult to give it an interpretation**
  - **used for ranking (≠ the null hypothesis tests)**

Mutual Information and Word Classes

The Problem

- Not enough data
  - **Language Modeling: we do not see “correct” n-grams**
    - – solution so far: smoothing
  - **suppose we see:**
    - – short homework, short assignment, simple homework
  - **but not:**
    - – simple assigment
  - **What happens to our (bigram) LM?**
    - – p(homework | simple) = high probability
    - – p(assigment | simple) = low probability (smoothed with p(assigment))
  - – They should be much closer!

Word Classes

- Observation: similar words behave in a similar way
  - trigram LM:
  - trigram LM, conditioning:
    - a ... homework (any atribute of homework: short, simple, late, difficult),
    - ... the woods (any verb that has the woods as an object: walk, cut, save)
  - trigram LM: both:
    - a (short,long,difficult,...) (homework,assignment,task,job,...)

Solution

- Use the Word Classes as the “reliability” measure
- Example: we see
    - **short homework, short assignment, simple homework**
  - but not:
    - **simple assigment**
  - Cluster into classes:
    - **(short, simple) (homework, assignment)**
      - covers “simple assignment”, too
- Gaining: realistic estimates for unseen n-grams
- Loosing: accuracy (level of detail) within classes

The New Model

- Rewrite the n-gram LM using classes:
  - Was: [k = 1..n]
    - **$p_k(w_i|h_i) = c(h_i,w_i) / c(h_i)$ [history: (k-1) words]**
  - Introduce classes:

$$p_k(w_i|h_i) = p(w_i|c_i)\, p_k(c_i|h_i)\ !$$

    - **history: classes, too: [for trigram: $h_i = c_{i-2},c_{i-1}$, bigram: $h_i = c_{i-1}$]**
  - Smoothing as usual
    - **over $p_k(w_i|h_i)$, where each is defined as above (except uniform which stays at 1/|V|)**

Training Data

there should be an equivalence

- Suppose we already have a mapping:
  - $r: V \rightarrow C$ assigning each word its class ($c_i = r(w_i)$)
- Expand the training data:
  - $T = (w_1, w_2, \ldots, w_{|T|})$ into
  - $T_C = (<w_1,r(w_1)>, <w_2,r(w_2)>, \ldots, <w_{|T|},r(w_{|T|})>)$
- Effectively, we have two streams of data:
  - word stream: $w_1, w_2, \ldots, w_{|T|}$
  - class stream: $c_1, c_2, \ldots, c_{|T|}$ (def. as $c_i = r(w_i)$)
- Expand Heldout, Test data too

Training the New Model

how many times did we see the word

how many times did it go to c_i

- As expected, using ML estimates:
  - $p(w_i|c_i) = p(w_i|r(w_i)) = c(w_i) / c(r(w_i)) = c(w_i) / c(c_i)$
    - **!!! $\mathbf{c(w_i,c_i) = c(w_i)}$ [since $\mathbf{c_i}$ determined by $\mathbf{w_i}$]**
  - $p_k(c_i|h_i)$:
    - $\mathbf{p_3(c_i|h_i) = p_3(c_i|c_{i-2}, c_{i-1}) = c(c_{i-2}, c_{i-1}, c_i) / c(c_{i-2}, c_{i-1})}$
    - $\mathbf{p_2(c_i|h_i) = p_2(c_i|c_{i-1}) = c(c_{i-1}, c_i) / c(c_{i-1})}$
    - $\mathbf{p_1(c_i|h_i) = p_1(c_i) = c(c_i) / |T|}$
- Then smooth as usual
  - not the $p(w_i|c_i)$ nor $p_k(c_i|h_i)$ individually, but the $\mathbf{p_k(w_i|h_i)}$

Classes: How To Get Them

- We supposed the classes are given
- Maybe there are in [human] dictionaries, but...
  - dictionaries are incomplete
  - dictionaries are unreliable
  - do not define classes as equivalence relation (overlap)
  - do not define classes suitable for LM
    - **small, short... maybe; small and difficult?**
- $\rightarrow$ we have to construct them <u>from data</u> (again...)

Creating the Word-to-Class Map

- We will talk about <u>bigrams</u> from now
- Bigram estimate:
  - $\mathbf{p_2(c_i|h_i) = p_2(c_i|c_{i-1}) = c(c_{i-1},c_i) / c(c_{i-1}) = c(r(w_{i-1}),r(w_i)) / c(r(w_{i-1}))}$
- Form of the model:
  - just raw bigram for now:
    - $\mathbf{P(T) = \prod_{i=1..|T|} p(w_i|r(w_i))\ p_2(r(w_i)|r(w_{i-1}))\quad (p_2(c_1|c_0) =_{df} p(c_1))}$
- Maximize over r (given r → fixed p, p_2): —> finding the best mapping
  - define objective $L(r) = 1/|T|\ \Sigma_{i=1..|T|} \log(p(w_i|r(w_i))\ p_2(r(w_i))|r(w_{i-1})))$
  - $r_{best} = \mathrm{argmax}_r\ L(r)$ (L(r) = norm. logprob of training data... as usual)

<— > this is what we are going to maximize in ML

Simplifying the Objective Function

- Start from $L(r) = 1/|T| \, \Sigma_{i=1..|T|} \log(p(w_i|r(w_i))\, p_2(r(w_i)|r(w_{i-1})))$:

p(a|b) · p(b) = p(a,b)

$1/|T| \, \Sigma_{i=1..|T|} \log(p(w_i|r(w_i))\, \underline{p(r(w_i))}\, p_2(r(w_i)|r(w_{i-1})) \,/\, \underline{p(r(w_i))}) =$

$1/|T| \, \Sigma_{i=1..|T|} \log(\underline{p(w_i,r(w_i))}\, p_2(r(w_i)|r(w_{i-1})) \,/\, p(r(w_i))) =$

clever simplification

$1/|T| \, \Sigma_{i=1..|T|} \log(\underline{p(w_i)}) + 1/|T| \, \Sigma_{i=1..|T|} \log(p_2(r(w_i)|r(w_{i-1})) \,/\, p(r(w_i))) =$

$-H(W) + 1/|T| \, \Sigma_{i=1..|T|} \log(p_2(r(w_i)|r(w_{i-1}))\, \underline{p(r(w_{i-1})) \,/\, (p(r(w_{i-1}))}\, p(r(w_i)))) =$

clever multiplication by 1

$-H(W) + 1/|T| \, \Sigma_{i=1..|T|} \log(\underline{p(r(w_i),r(w_{i-1}))} \,/\, (p(r(w_{i-1}))\, p(r(w_i)))) =$

$-H(W) + \Sigma_{d,e \in C}\, p(d,e) \log(\, p(d,e) \,/\, (p(d)\, p(e))\,) =$

$-H(W) + I(D,E)$

(event E picks class adjacent (to the right) to the one picked by D)

- Since W does not depend on r, we ended up with **the need to maximize** I(D,E).

This is way more simplified

Maximizing Mutual Information
(dependent on the mapping r)

- Result from previous foil:
  - Maximizing the probability of data amounts to maximizing I(D,E), the mutual information of the adjacent classes.
- Good:
  - We know what a MI is, and we know how to maximize.
- Bad:
  - There is no way how to maximize over so many possible partitionings: $|V|^{|V|}$ - no way to test them all.

$|C|^{|V|}$ possible allignments into |C| classes.

This would be impossible to compute by brute-force

Training or Heldout?

- Training:
  - best I(D,E): all words in a class of its own
    - → **will not give us anything new.**
- Heldout: ok, but:
  - must smooth to test any possible partitioning (unfeasible):
    - → **using raw model: 0 probability of heldout (almost) guaranteed**
      - → will not be able to compare anything
  - some smoothing estimates? (to be explored...)
- Solution:
  - use training anyway, but only keep I(D,E) as large as possible

We wouldn't compute the alg. Only might think about the outcome or description

The Greedy Algorithm

→ when we want to merge two (similar) classes

- Define merging operation on the mapping r: $V \rightarrow C$:
  - merge: $R \times C \times C \rightarrow R' \times C^{-1}$: $(r,k,l) \rightarrow r',C'$ such that
  - $C^{-1} = \{C - \{k,l\} \cup \{m\}\}$ (throw out k and l, add new $m \notin C$)
  - $r'(w) =$ ..... m for $w \in r_{INV}(\{k,l\})$,

    ..... r(w) otherwise.
- 1. Start with each word in its own class (C = V), r = id.
- 2. Merge two classes k,l into one, m, such that

  $$(k,l) = \text{argmax}_{k,l} I_{merge(r,k,l)}(D,E).$$

  → gives at most V^2 combinations
- 3. Set new (r,C) = merge(r,k,l). → each step decreases classes by one
- 4. Repeat 2 and 3 until |C| reaches predetermined size.

Word Classes in Applications

- Word Sense Disambiguation: context not seen [enough(-times)]
- Parsing: verb-subject, verb-object relations
- Speech recognition (acoustic model): need more instances of [rare(r)] sequences of phonemes
- Machine Translation: translation equivalent selection [for rare(r) words]

Word Classes:
Programming Tips & Tricks

The Algorithm (review)

- Define merge(r,k,l) = (r’,C’) such that
  - **C’ = C - {k,l} ∪ {m (a new class)}**
  - **r’(w) = r(w) except for k,l member words for which it is m.**
- 1. Start with each word in its own class (C = V), r = id.
- 2. Merge two classes k,l into one, m, such that

$$(k,l) = \text{argmax}_{k,,l}\ I_{merge(r,k,l)}(D,E).$$

- 3. Set new (r,C) = merge(r,k,l).
- 4. Repeat 2 and 3 until |C| reaches a predetermined size.

Complexity Issues

- Still too complex:
  - |V| iterations of the steps 2 and 3.
  - $|V|^2$ steps to maximize $\text{argmax}_{k,l}$ (selecting k,l freely from |C|, which is in the order of $|V|^2$)
  - $|V|^2$ steps to compute I(D,E) (sum within sum, all classes, also: includes log)
  - $\Rightarrow$ total: $|V|^5$
  - i.e., for |V| = 100, about 10^{10} steps; ~ several hours!
  - but |V| ~ 50,000 or more

Trick #1: Recomputing The MI the Smart Way: Subtracting...

- Bigram count table:

l \ r	c_1	c_2	c_3	c_4
c_1	10	2	0	1
c_2	0	0	5	2
c_3	0	2	0	3
c_4	2	3	0	0

- Test-merging c_2 and c_4: recompute only rows/cols 2 & 4:
  - subtract column/row (2 & 4) from the MI sum (intersect.!)
  - add sums of merged counts (row & column)

...and Adding

- Add the merged counts:

l \ r	c_1	c_2'	c_3
c_1	10	3	0
c_2'	2	5	5
c_3	0	5	0

- Be careful at intersections:
  - (don't forget to add this:)

	c_2	c_3	c_4
c_2	0	5	2
c_3	2	0	3
c_4	3	0	0

Trick #2: Precompute the Counts-to-be-Subtracted

- Summing loop goes through i,j
- ...but the single row/column sums do not depend on the (resulting sums after the) merge
- $\Rightarrow$ can be precomputed
    - **only 2k logs to compute at each algorithm iteration, instead of k^2**
- Then for each "merge-to-be" compute only add-on sums, plus "intersection adjustment"

Formulas for Tricks #1 and #2

- Let's have k classes at a certain iteration. Define:

$$q_k(l,r) = p_k(l,r) \log(p_k(l,r) / (p_{kl}(l)\, p_{kr}(r)))$$

now the same, but using counts:

$$q_k(l,r) = c_k(l,r)/N \log(N\, c_k(l,r)/(c_{kl}(l)\, c_{kr}(r)))$$

- Define further (row+column i sum):

intersection adjustment

precomputed

$$s_k(a) = \Sigma_{l=1..k} q_k(l,a) + \Sigma_{r=1..k} q_k(a,r) - q_k(a,a)$$

- Then, the subtraction part of Trick #1 amounts to

$$sub_k(a,b) = s_k(a) + s_k(b) - q_k(a,b) - q_k(b,a)$$

remaining intersect. adj.

Formulas - cont.

- After-merge add-on:

$$add_k(a,b) = \Sigma_{l=1..k,l\neq a,b} q_k(l,a+b) + \Sigma_{r=1..k,r\neq a,b} q_k(a+b,r) + q_k(a+b,a+b)$$

- What is it a+b? Answer: the new (merged) class.
- Hint: use the definition of q_k as a “macro”, and then

$$p_k(a+b,r) = p_k(a,r) + p_k(b,r) \text{ (same for other sums, equivalent)}$$

- The above sums cannot be precomputed
- After-merge Mutual Information (I_k is the “old” MI, kept from previous iteration of the algorithm):

$$I_k(a,b) \text{ (MI after merge of cl. a,b)} = I_k - sub_k(a,b) + add_k(a,b)$$

Trick #3: Ignore Zero Counts

- Many bigrams are 0
  - (see the paper: Canadian Hansards, < .1 % of bigrams are non-zero)
- Create linked lists of non-zero counts in columns and rows (similar effect: use perl's hashes)
- Update links after merge (after step 3)

Trick #4: Use Updated Loss of MI

- We are now down to $|V|^4$: |V| merges, each merge takes $|V|^2$ "test-merges", each test-merge involves order-of-|V| operations ($add_k(i,j)$ term, foil #8)
- *Observation*: many numbers (s_k, q_k) needed to compute the mutual information loss due to a merge of i+j ***do not change***: namely, those which are not in the vicinity of neither i nor j.
- *Idea*: keep the MI loss matrix for all pairs of classes, and (after a merge) update only those cells which have been influenced by the merge.

Formulas for Trick #4 (s_{k-1}, L_{k-1})

- Keep a matrix of "losses" $L_k(d,e)$.[1]
- Init: $L_k(d,e) = sub_k(d,e) - add_k(d,e)$ [then $I_k(d,e) = I_k - L_k(d,e)$]
- Suppose a,b are now the two classes merged into a:
- Update (k-1: index used for the *next* iteration; $i,j \neq a,b$):
  - $s_{k-1}(i) = s_k(i) - q_k(i,a) - q_k(a,i) - q_k(i,b) - q_k(b,i) + q_{k-1}(a,i) + q_{k-1}(i,a)$
  - [2]$L_{k-1}(i,j) = L_k(i,j) - s_k(i) + s_{k-1}(i) - s_k(j) + s_{k-1}(j) +$

    $+ q_k(i+j,a) + q_k(a,i+j) + q_k(i+j,b) + q_k(b,i+j) -$

    $- q_{k-1}(i+j,a) - q_{k-1}(a,i+j)$ *[NB: may substitute even for s_k , s_{k-1}]*

NB [1] L_k is symmetrical $L_k(d,e) = L_k(e,d)$ (q_k is something different!)

[2]The update formula $L_{k-1}(l,m)$ is wrong in the Brown et. al paper

Completing Trick #4

- $s_{k-1}(a)$ must be computed using the "Init" sum.
- $L_{k-1}(a,i) = L_{k-1}(i,a)$ must be computed in a similar way, for all $i \neq a,b$.
- $s_{k-1}(b)$, $L_{k-1}(b,i)$, $L_{k-1}(i,b)$ are not needed anymore (keep track of such data, i.e. mark every class already merged into some other class and do not use it anymore).
- Keep track of the minimal loss during the $L_k(i,j)$ update process (so that the next merge to be taken is obvious immediately after finishing the update step).

Efficient Implementation

- Data Structures: (N - # of bigrams in data [fixed])
  - Hist(k) history of merges
    - **Hist(k) = (a,b) merged when the remaining number of classes was k**
  - $c_k(i,j)$ bigram class counts [updated]
  - $c_{kl}(i)$, $c_{kr}(i)$ unigram (marginal) counts [updated]
  - $L_k(a,b)$ table of losses; upper-right trianlge [updated]
  - $s_k(a)$ "subtraction" subterms [optionally updated]
  - $q_k(i,j)$ subterms involving a log [opt. updated]
    - **The optionally updated data structures will give linear improvement only in the subsequent steps, but at least $s_k(i)$ is necessary in the initialization phase (1st iteration)**

Implementation: the Initialization Phase

- 1 Read data in, init counts $c_k(l,r)$; then $\forall l,r,a,b;\ a < b$:
- 2 Init unigram counts:

$$c_{kl}(l) = \Sigma_{r=1..k} c_k(l,r), \qquad c_{kr}(r) = \Sigma_{l=1..k} c_k(l,r)$$

  - complicated? remember, must take care of start & end of data!
- 3 Init $q_k(l,r)$: use the 2nd formula (count-based) on foil 7,

$$q_k(l,r) = c_k(l,r)/N \log(N\, c_k(l,r)/(c_{kl}(l)\, c_{kr}(r)))$$

- 4 Init $s_k(a) = \Sigma_{l=1..k} q_k(l,a) + \Sigma_{r=1..k} q_k(a,r) - q_k(a,a)$
- 5 Init $L_k(a,b) = s_k(a)+s_k(b)-q_k(a,b)-q_k(b,a)-q_k(a+b,a+b)+$

$$- \Sigma_{l=1..k,l\neq a,b} q_k(l,a+b) - \Sigma_{r=1..k,r\neq a,b} q_k(a+b,r)$$

Implementation: Select & Update

- 6 Select the best pair (a,b) to merge into a (watch the candidates when computing $L_k(a,b)$); save to Hist(k)
- 7 Optionally, update $q_k(i,j)$ for all $i,j \neq b$, get $q_{k-1}(i,j)$
  - remember those $q_k(i,j)$ values needed for the updates below
- 8 Optionally, update $s_k(i)$ for all $i \neq b$, to get $s_{k-1}(i)$
  - again, remember the $s_k(i)$ values for the "loss table" update
- 9 Update the loss table, $L_k(i,j)$, to $L_{k-1}(i,j)$, using the tabulated q_k, q_{k-1}, s_k and s_{k-1} values, or compute the needed $q_k(i,j)$ and $q_{k-1}(i,j)$ values dynamically from the counts: $c_k(i+j,b) = c_k(i,b) + c_k(j,b)$; $c_{k-1}(a,i) = c_k(a+b,i)$

Towards the Next Iteration

- 10 During the $L_k(i,j)$ update, keep track of the minimal loss of MI, and the two classes which caused it.
- 11 Remember such best merge in Hist(k).
- 12 Get rid of all s_k, q_k, L_k values.
- 13 Set k = k -1; stop if k == 1.
- 14 Start the next iteration
  - either by the optional updates (steps 7 and 8), or
  - directly updating $L_k(i,j)$ again (step 9).

Moving Words Around

- Improving Mutual Information
  - take a word from one class, move it to another (i.e., two classes change: the moved-from and the moved-to), compute $I_{new}(D,E)$; keep change permanent if

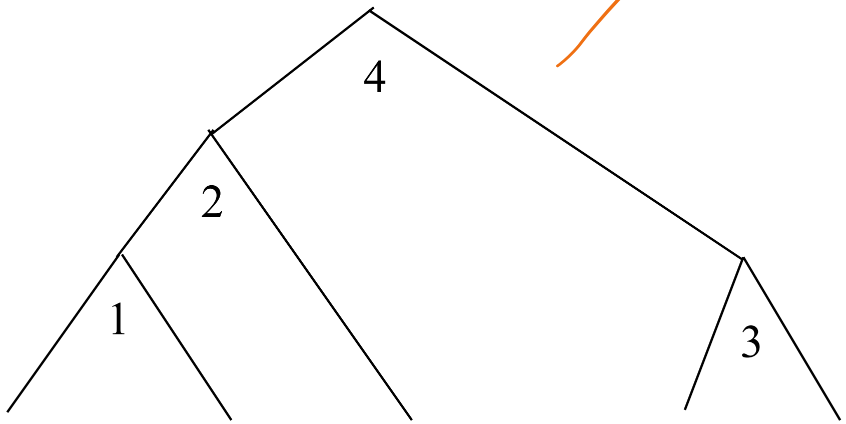
    $$I_{new}(D,E) > I(D,E)$$

  - keep moving words until no move improves I(D,E)
- Do it at every iteration, or at every m iterations
- Use similar "smart" methods as for merging

Using the Hierarchy

- Natural Form of Classes
  - follows from the sequence of merges:

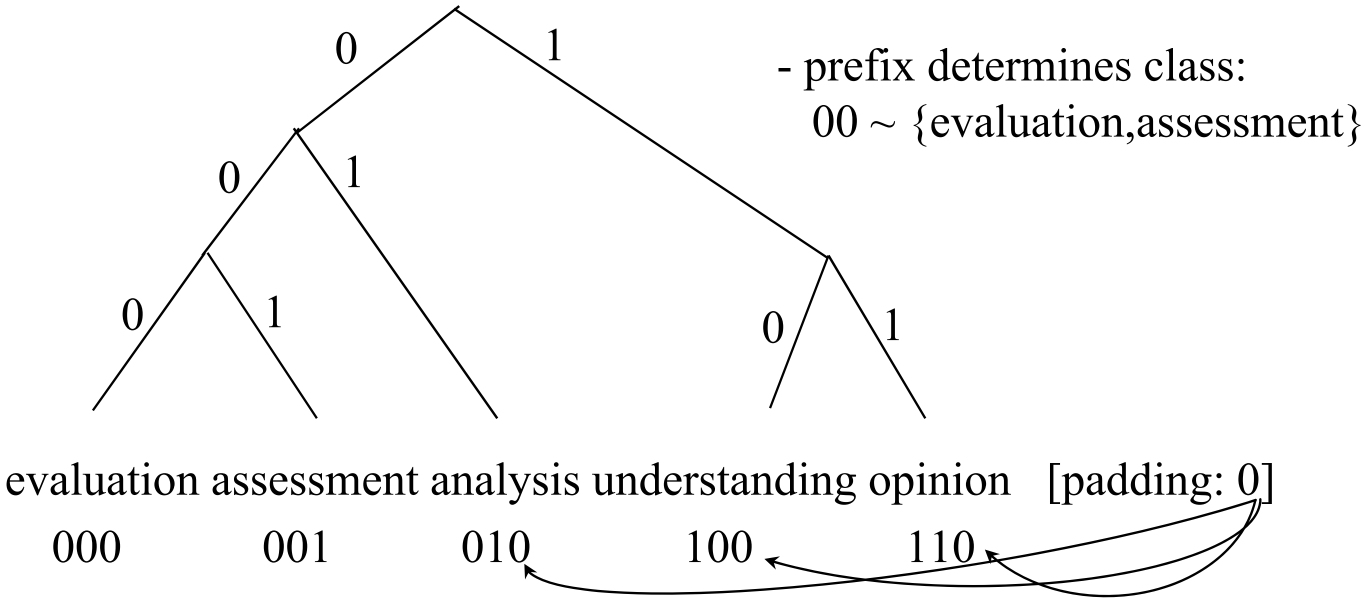
you dont actually need to go deep to the leaf all the time



evaluation assessment analysis understanding opinion

Numbering the Classes (within the Hierarchy)

- Binary branching
- Assign 0/1 to the left/right branch at every node:



Markov Models

Review: Markov Process

- Bayes formula (chain rule):

we don't consider whole history

$$P(W) = P(w_1,w_2,\dots,w_T) = \Pi_{i=1..T}\, p(w_i|w_1,w_2,..,w_{i-n+1},..,w_{i-1})$$

approximation

- n-gram language models:
  - Markov process (chain) of the order n-1:

$$P(W) = P(w_1,w_2,\dots,w_T) = \Pi_{i=1..T}\, p(w_i|w_{i-n+1},w_{i-n+2},..,w_{i-1})$$

Using just <u>one</u> distribution (Ex.: trigram model: $p(w_i|w_{i-2},w_{i-1})$):

Positions:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Words:	My	car	**broke**	**down**	**,**	and	within	hours	Bob	's	car	**broke**	**down**	**,**	too	.

p(**,|broke down**) = $p(w_5|w_3,w_4)) = p(w_{14}|w_{12},w_{13})$

Markov Properties

- Generalize to any process (not just words/LM):
  - Sequence of random variables: $X = (X_1, X_2, \ldots, X_T)$
  - Sample space S (*states*), size N: $S = \{s_0, s_1, s_2, \ldots, s_N\}$

1. Limited History (Context, Horizon):

$$\forall i \in 1..T;\ P(X_i | X_1, \ldots, X_{i-1}) = P(X_i | X_{i-1})$$

1 7 3 7 9 0 6 7 3 4 5... generalization 1 7 3 7 9 0 6 7 3 4 5...

2. Time invariance (M.C. is stationary, homogeneous)

$$\forall i \in 1..T,\ \forall y,x \in S;\ P(X_i = y | X_{i-1} = x) = p(y|x)$$

1 7 3 7 9 0 6 7 3 4 5...

? ok...same ***distribution***

Long History Possible

- What if we want trigrams:

  1 7 3 7 9 0 6 7 3 4 5...

- Formally, use transformation:

  Define new variables Q_i, such that $X_i = \{Q_{i-1},Q_i\}$:

  Then

  $$P(X_i|X_{i-1}) = P(Q_{i-1},Q_i|Q_{i-2},Q_{i-1}) = P(Q_i|Q_{i-2},Q_{i-1})$$

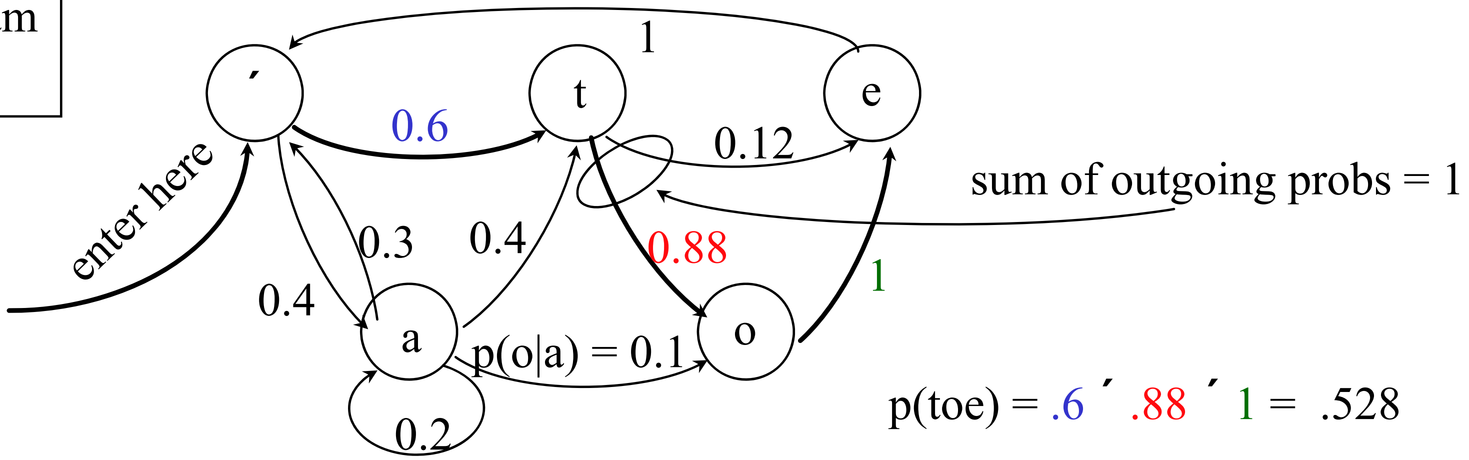
  Just technical thing

Predicting (X_i):	1 7 3 7 9 0 6 7 3 9 0...
	× 1 7 3 .... 0 6 7 3 4 0
History ($X_{i-1} = \{Q_{i-2},Q_{i-1}\}$):	× × 1 7 .... 9 0 6 7 3 9

Graph Representation: State Diagram

- $S = \{s_0, s_1, s_2, \ldots, s_N\}$: states
- Distribution $P(X_i|X_{i-1})$:
  - **transitions (as arcs) with probabilities attached to them:**

Bigram case:

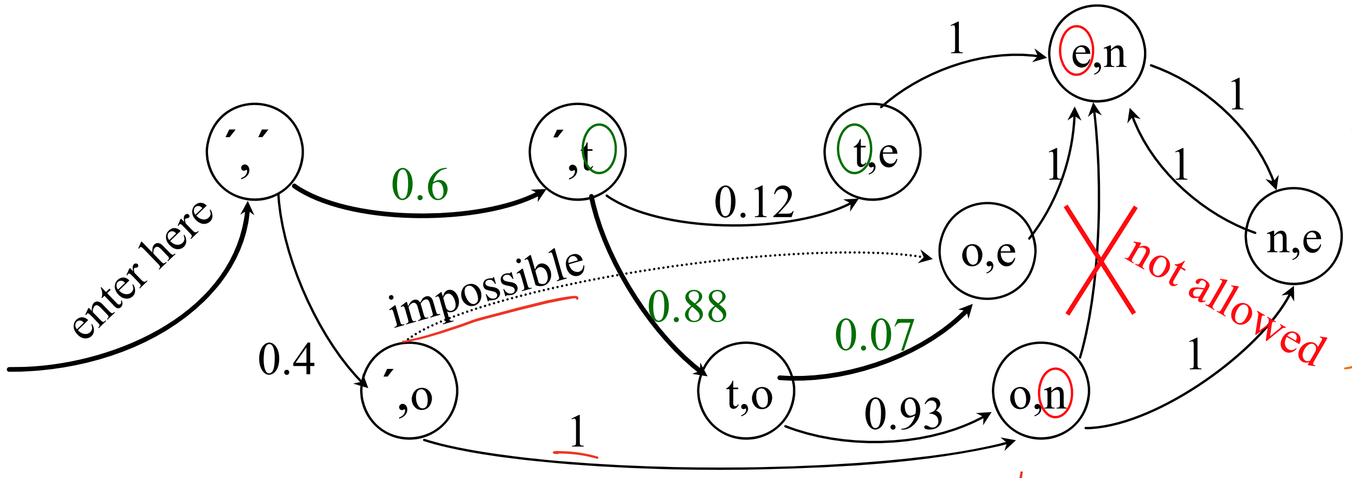


p(toe) = .6 ´ .88 ´ 1 = .528

The Trigram Case

- $S = \{s_0, s_1, s_2, \ldots, s_N\}$: states: pairs $s_i = (x,y)$
- Distribution $P(X_i|X_{i-1})$: (r.v. X: generates pairs s_i)

the chain must be compatible:
o,e -> e,n
o,n -/-> e,n



p(toe) = .6 × .88 × .07 ≅ .037

since one arc has prob. 1, there can't be any other

p(one) = ?

this doesn't simulate „hidden" features of language very well...

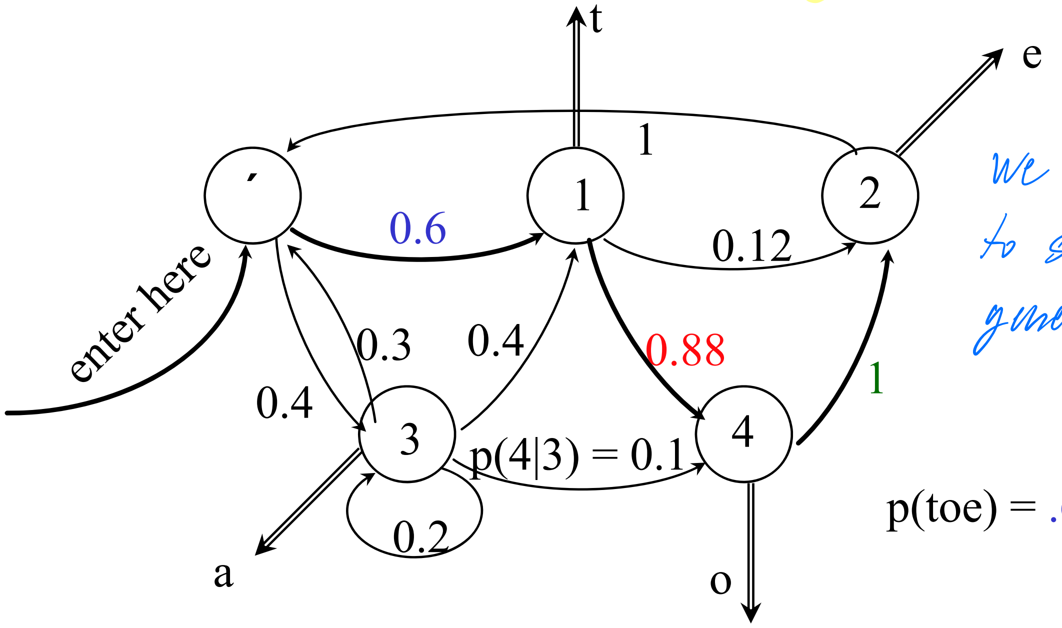
Finite State Automaton

- States ~ symbols of the [input/output] alphabet
  - pairs (or more): last element of the n-tuple
- Arcs ~ transitions (sequence of states)
- [Classical FSA: alphabet symbols on arcs:
  - transformation: arcs ↔ nodes]
- Possible thanks to the "limited history" M'ov Property
- So far: <u>***Visible***</u> Markov Models (VMM)

allows the states to represent anything

Hidden Markov Models

- The simplest HMM: states generate [observable] output (using the "data" alphabet) but remain "invisible":

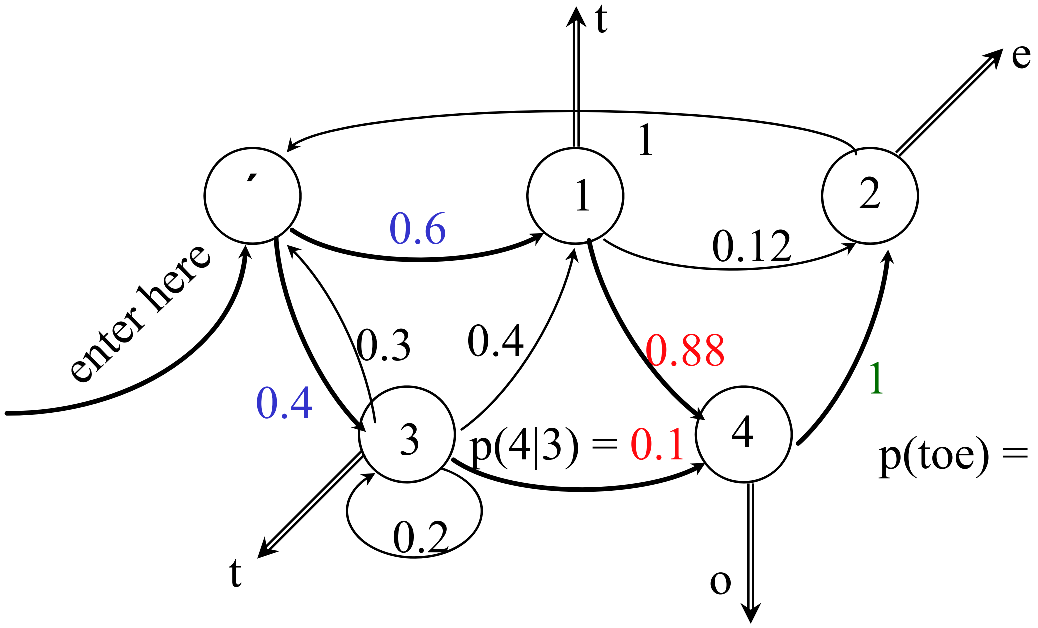


we need to go to states that generate the required items.

p(toe) = .6 ´ .88 ´ 1 = .528

Added Flexibility

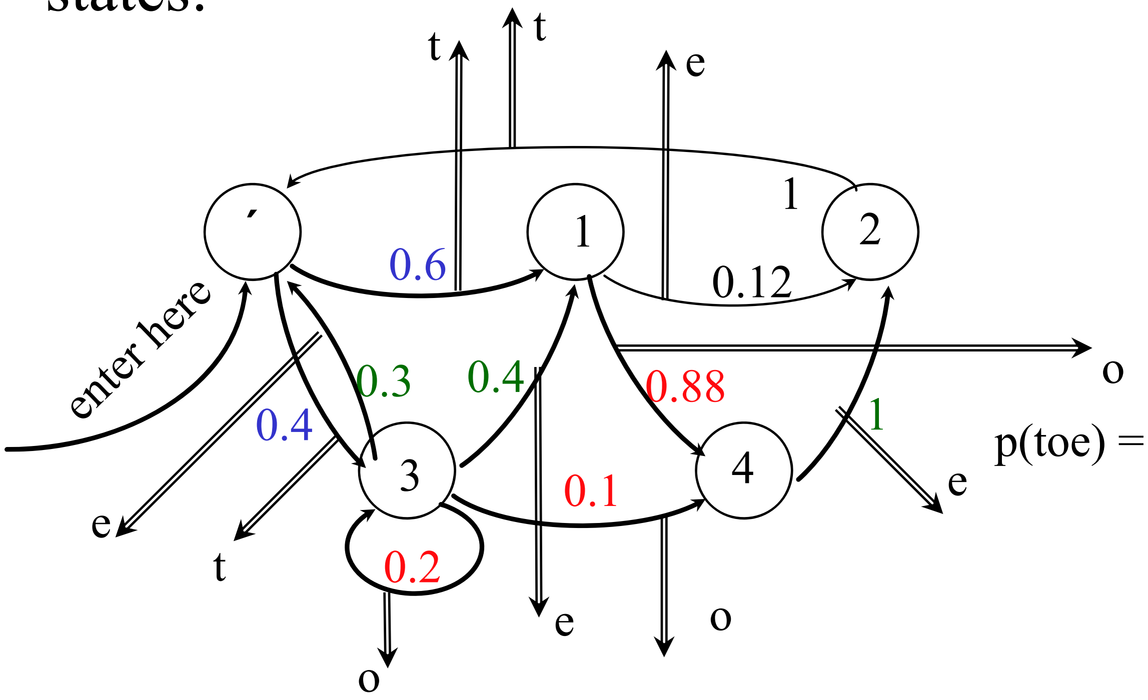
- So far, no change; but different states may generate the same output (why not?):



p(toe) = .6 ´ .88 ´ 1 +
.4 ´ .1 ´ 1 = .568

Output from Arcs...

- Added flexibility: Generate output from arcs, not states:



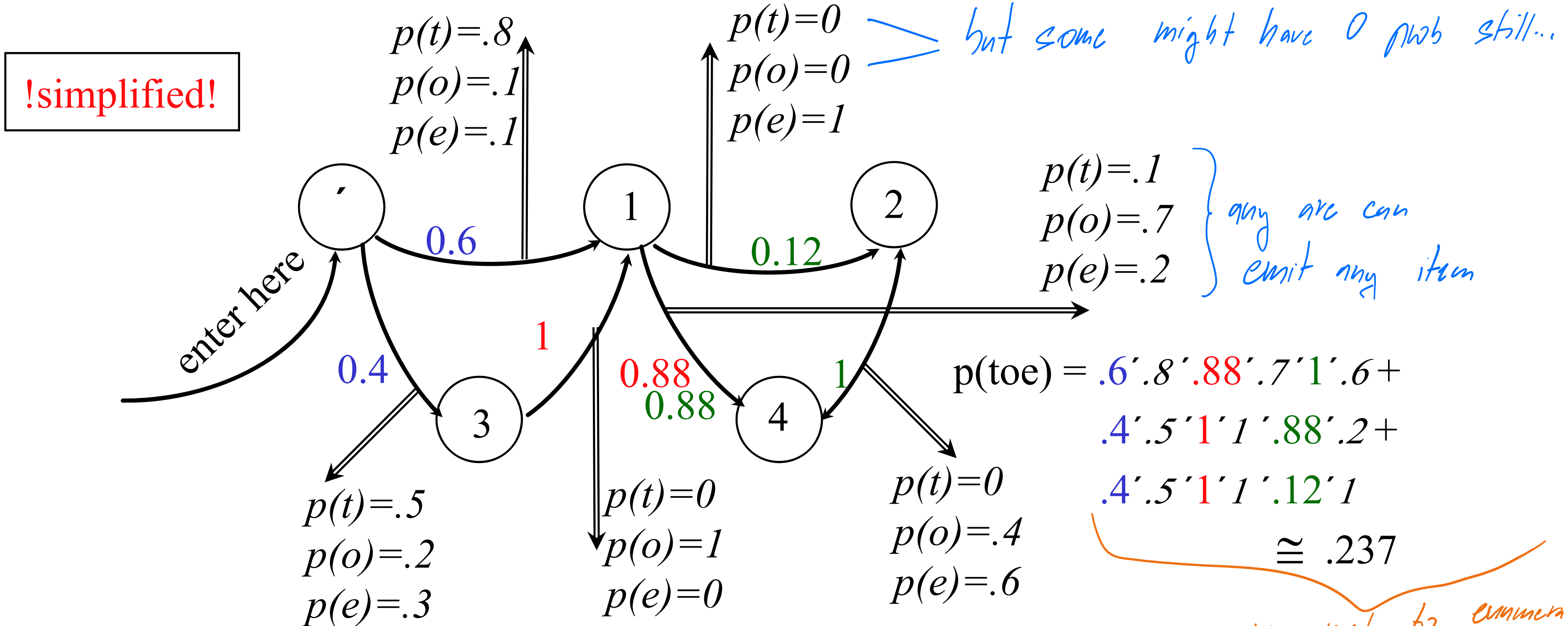
we need to enumerate over all possible ways in the graph

p(toe) = .6 ´ .88 ´ 1 +
.4 ´ .1 ´ 1 +
.4 ´ .2 ´ .3 +
.4 ´ .2 ´ .4 = .624

Ve skutečnosti generuje symbol ta dvojice stavů S_{i-1}, S_i, ale ta jednoznačně určuje tu hranu, takže ho to jde korelovat i tak že hrana generuje symbol.

... and Finally, Add Output Probabilities

- Maximum flexibility: [Unigram] distribution (sample space: output alphabet) at each output arc:



$$p(toe) = .6 \times .8 \times .88 \times .7 \times 1 \times .6 + .4 \times .5 \times 1 \times 1 \times .88 \times .2 + .4 \times .5 \times 1 \times 1 \times .12 \times 1 \cong .237$$

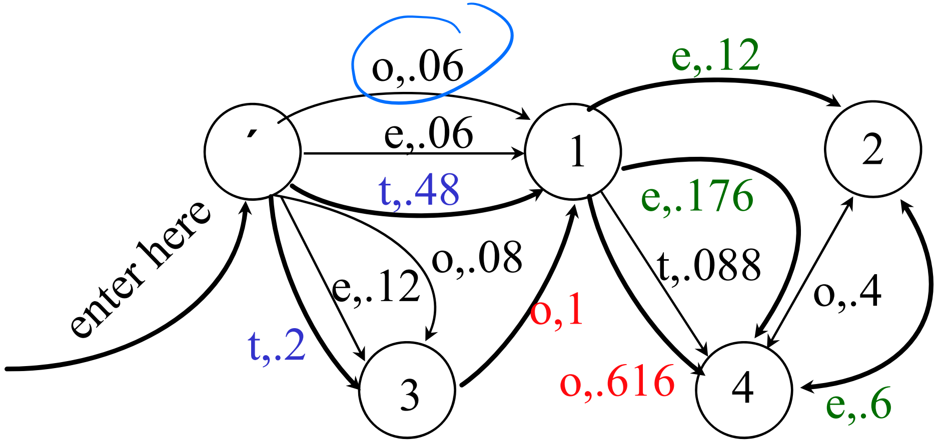
we need to enumerate all possible paths

This is easier to find all possible paths that allign with the input sequence.

Slightly Different View

- Allow for multiple arcs from $s_i \rightarrow s_j$, mark them by output symbols, get rid of output distributions:

$= p(x_0 \to x_1) \cdot p(o)$ → combination of both probability distributions



p(toe) = .48´.616´.6+ .2´1´.176 + .2´1´.12 ≅ .237

In the future, we will use the view more convenient for the problem at hand.

describe what HMMs are good for, think of their description
no computation again

Formalization

- HMM (the most general case):
  - five-tuple (S, s_0, Y, P_S, P_Y), where:
    - **$S = \{s_0,s_1,s_2,\ldots,s_T\}$ is the set of states, s_0 is the initial state,**
    - **$Y = \{y_1,y_2,\ldots,y_V\}$ is the output alphabet,** —> must be known ahead
    - **$P_S(s_j|s_i)$ is the set of prob. distributions of transitions,**
      - size of P_S: $|S|^2$. —> all possible pairs of states
    - **$P_Y(y_k|s_i,s_j)$ is the set of output (emission) probability distributions.**
      - size of P_Y: $|S|^2 \times |Y|$ —> all possible pairs of states combined with all possible letters
- Example:
  - $S = \{x, 1, 2, 3, 4\}$, $s_0 = x$
  - $Y = \{ t, o, e \}$

Formalization - Example

- Example (for graph, see foils 11,12):
  - $S = \{x, 1, 2, 3, 4\}$, $s_0 = x$
  - $Y = \{ e, o, t \}$
  - P_S:

transition probability:

rows must sum to 1 →

	x	1	2	3	4
x	0	.6	0	.4	0
1	0	0	.12	0	.88
2	0	0	0	0	1
3	0	1	0	0	0
4	0	0	1	0	0

$\Sigma = 1$

I can only go to „4" from „2".

- P_Y:

probability of generating „t" when doing transition from state i → j

t	x	1	2	3	4
x		.8		.5	
1			0		.1
2					0
3		0			
4			0		

o	x	1	2	3	4
x					.7

e	x	1	2	3	4
x					.2

$\Sigma = 1$

Using the HMM

- The generation algorithm (of limited value :-)):
  1. **Start in $s = s_0$.**
  2. **Move from s to s' with probability $P_S(s'|s)$.**
  3. **Output (emit) symbol y_k with probability $P_S(y_k|s,s')$.**
  4. **Repeat from step 2 (until somebody says enough).**
- More interesting usage:
  - Given an output sequence $Y = \{y_1,y_2,\ldots,y_k\}$, compute its probability.
  - Given an output sequence $Y = \{y_1,y_2,\ldots,y_k\}$, compute the most likely sequence of states which has generated it.
  - ...plus variations: e.g., n best state sequences

HMM Algorithms: Trellis and Viterbi

HMM: The Two Tasks

- HMM (the general case):
  - five-tuple (S, S_0, Y, P_S, P_Y), where:
    - **$S = \{s_1,s_2,\ldots,s_T\}$ is the set of states, S_0 is the initial state,**
    - **$Y = \{y_1,y_2,\ldots,y_V\}$ is the output alphabet,**
    - **$P_S(s_j|s_i)$ is the set of prob. distributions of transitions,**
    - **$P_Y(y_k|s_i,s_j)$ is the set of output (emission) probability distributions.**
- Given an HMM & an output sequence $Y = \{y_1,y_2,\ldots,y_k\}$:

  (Task 1) compute the probability of Y; → I have two sequences and want to know which one is more likely to happen

  (Task 2) compute the most likely sequence of states which has generated Y. „decode" the path

This is case of dynamic programming

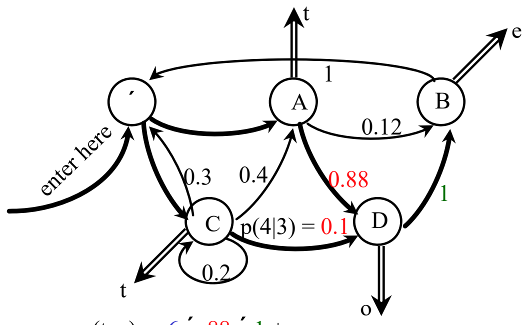
$O(m \cdot t)$ size

The prefix (and his probability) will never change.

Trellis - Deterministic Output

For word of len 3, we need 3 transitions.

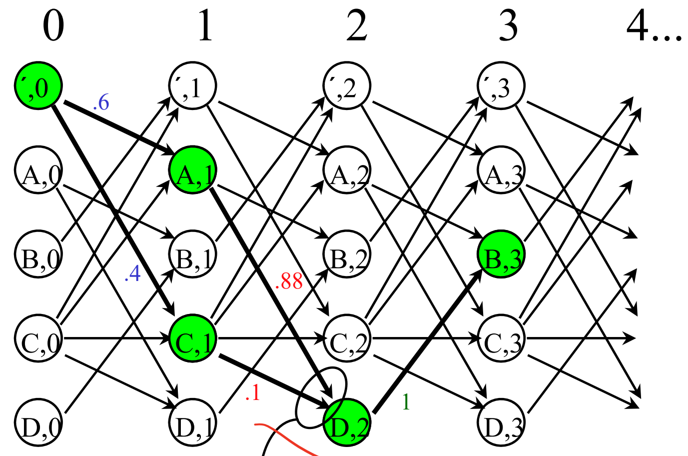
HMM:



p(toe) = .6 ´ .88 ´ 1 +
.4 ´ .1 ´ 1 = .568

"rollout"

Trellis: time/position t



doesn't need to sum up to 1.

Y: t + o e

α(´,0) = 1 α(A,1) = .6 α(D,2) = .568 α(B,3) = .568

α(C,1) = .4

the sum of two possible paths

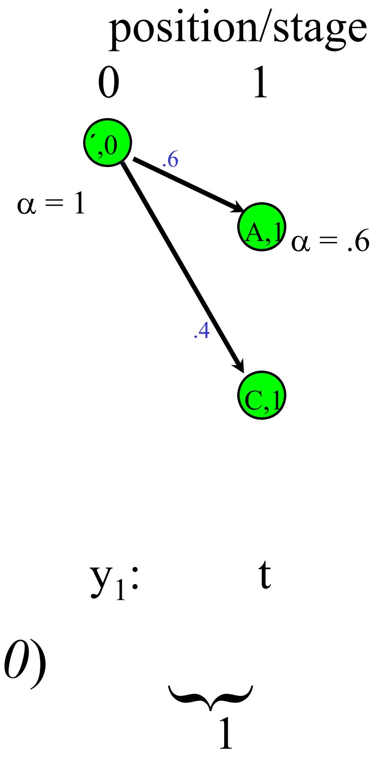
there was only one arc with prob. == 1

- trellis state: (HMM state, position)
- each state: holds ***one*** number (prob): α
- probability or Y: Σα in the last state

Creating the Trellis: The Start

- Start in the start state (×),
  - set its α(×,*0*) to 1.
- Create the first stage:
  - get the first "output" symbol y_1
  - create the first stage (column)
  - but only those trellis states

    which generate y_1
  - set their α(*state*,*1*) to the P_S(*state*|×) α(×,*0*)
- ...and forget about the *0*-th stage



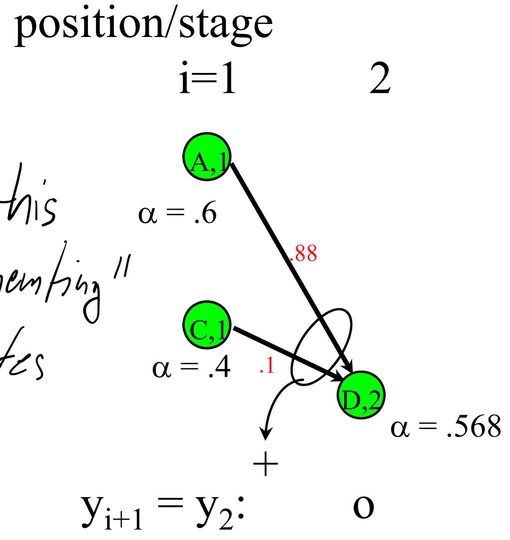
Trellis: The Next Step

- Suppose we are in stage *i*
- Creating the next stage:
  - create all trellis states in the next stage which generate y_{i+1}, but only those reachable from any of the stage-*i* states
  - set their $\alpha(state,i+1)$ to:

    $P_S(state|prev.state) \times \alpha(prev.state, i)$

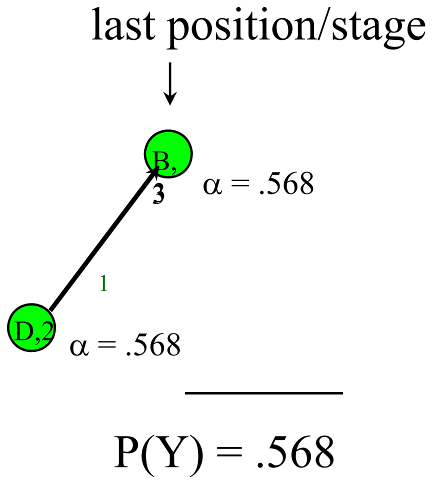
    (add up all such numbers on arcs going to a common trellis state)
  - ...and forget about stage *i*

we keep this for all "generating" states



Trellis: The Last Step

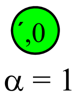
- Continue until "output" exhausted
  - |Y| = 3: until stage 3
- Add together all the α(*state*,|*Y*|)
- That's the <u>P(Y)</u>.
- Observation (pleasant): we only need two steps
  - memory usage max: 2|S|
  - multiplications max: $|S|^2|Y|$

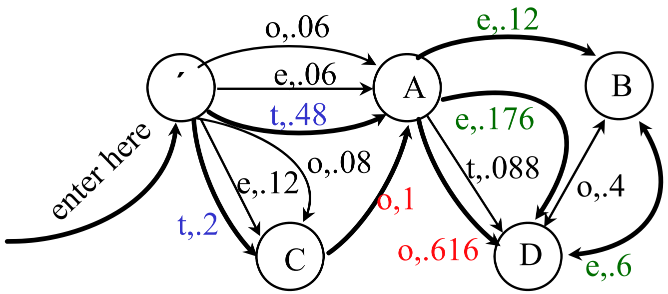


We can end up having more final states. We would sum their probs together.

Trellis: The General Case (still, bigrams)

- Start as usual:
  - start state (´), set its α(´,*0*) to 1.





... now edges are generating output

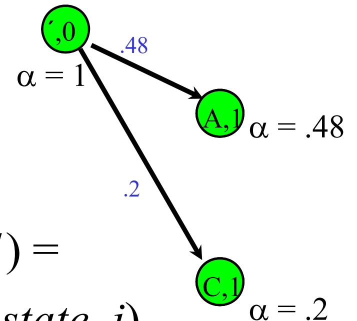
p(toe) = .48´.616´.6+
.2´1´.176 +
.2´1´.12 ≅ .237

General Trellis: The Next Step

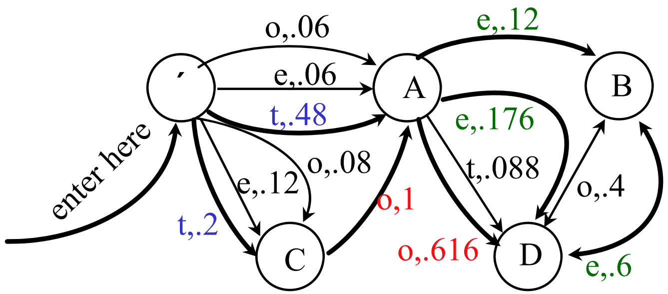
- We are in stage *i* :
  - Generate the next stage *i+1* as before (except now arcs generate output, thus use only those arcs marked by the output symbol y_{i+1})
  - For each generated *state*, compute $\alpha(state, i+1) =$
    $= \Sigma_{\text{incoming arcs}} P_Y(y_{i+1}|state, prev.state) \times \alpha(prev.state, i)$

position/stage

0 1



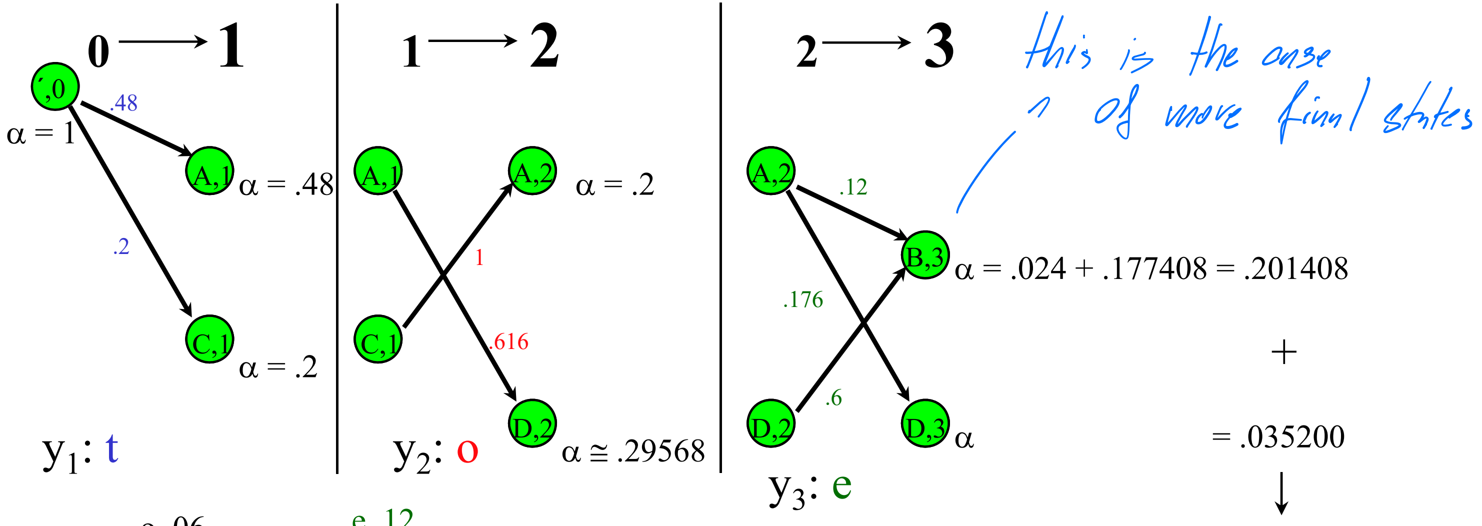
y_1: t



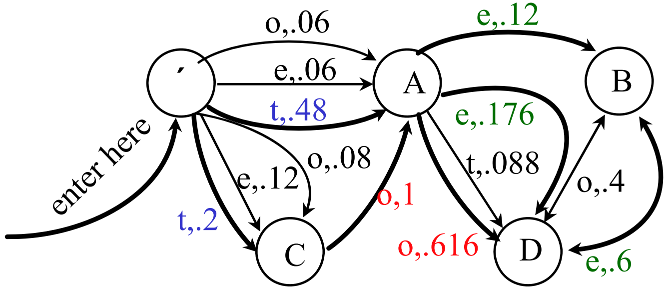
...and forget about stage *i* as usual.

Trellis: The Complete Example

Stage:

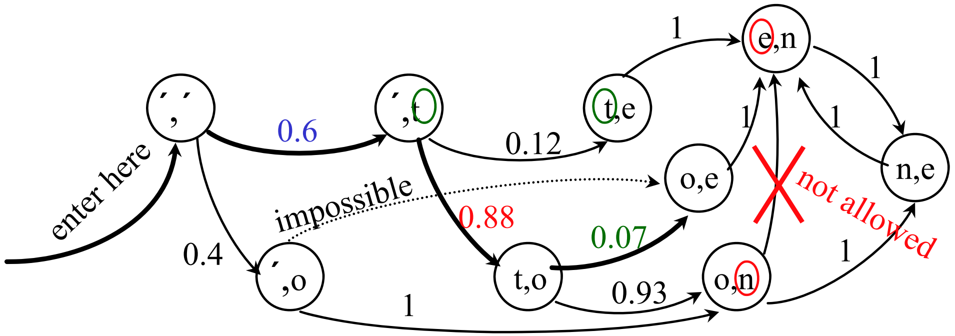


$P(Y) = P(toe) = .236608$

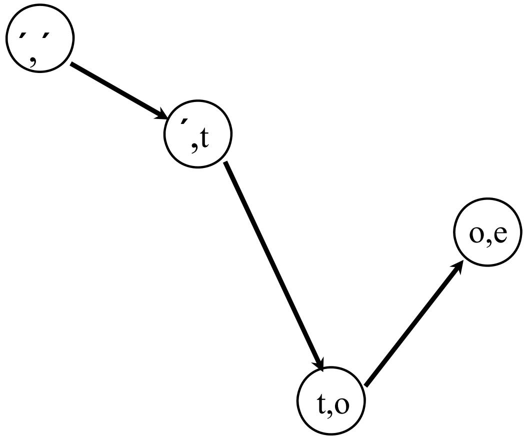


The Case of Trigrams

- Like before, but:
  - states correspond to bigrams,
  - output function always emits the second output symbol of the pair (state) to which the arc goes:



p(toe) = .6 ´ .88 ´ .07 ≅ .037



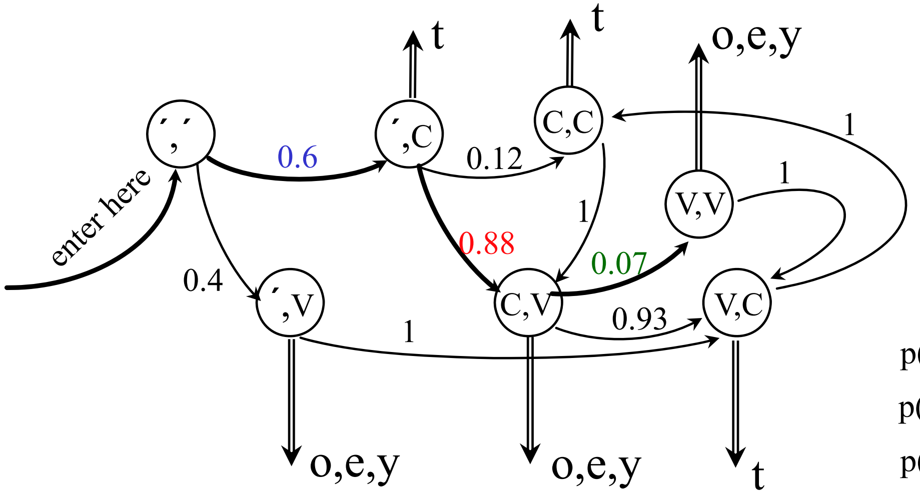
Multiple paths not possible → trellis not really needed

Trigrams with Classes

- More interesting:
  - n-gram class LM: $p(w_i|w_{i-2},w_{i-1}) = p(w_i|c_i)\ p(c_i|c_{i-2},c_{i-1})$

    → states are pairs of classes (c_{i-1},c_i), and emit “words”:

    (letters in our example)

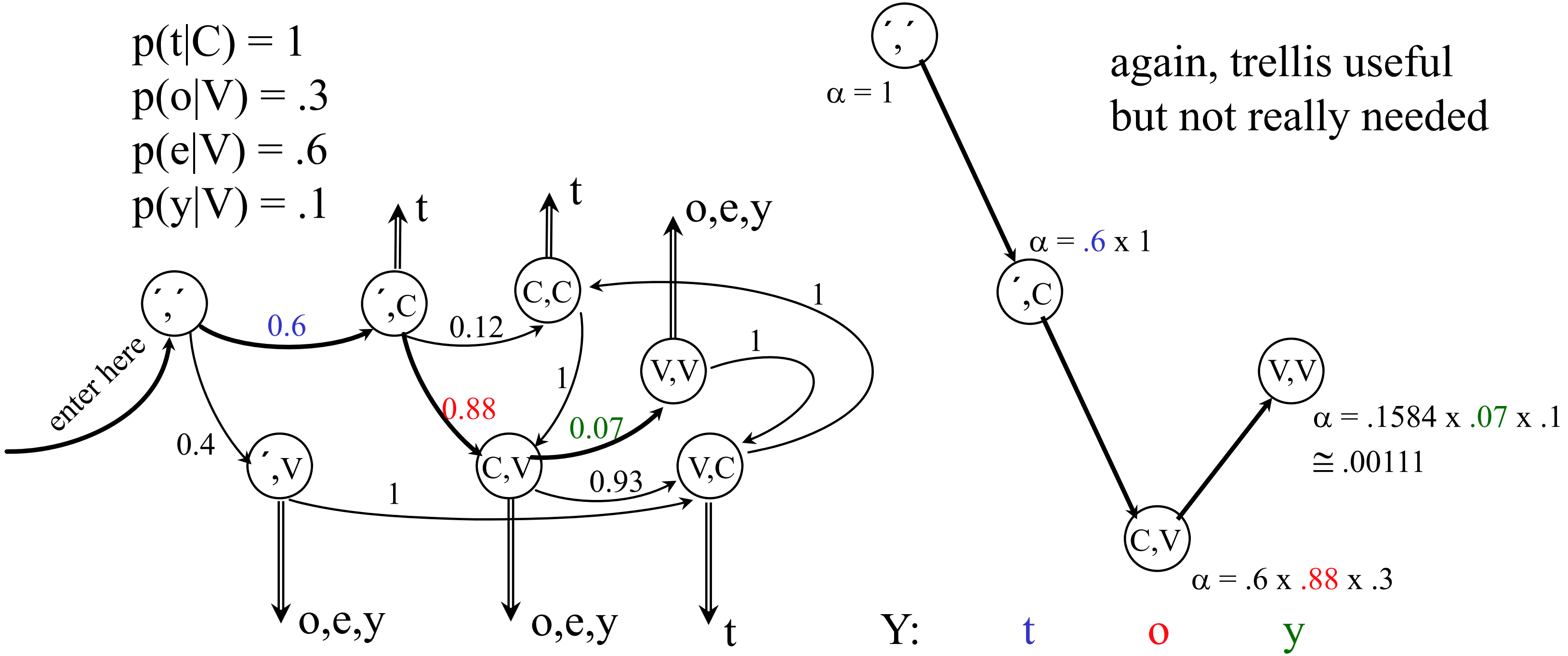


$p(t|C) = 1$ usual,
$p(o|V) = .3$ non-
$p(e|V) = .6$ overlapping
$p(y|V) = .1$ classes

p(toe) = .6 ´ 1 ´ .88 ´ .3 ´ .07 ´ .6 ≅ .00665
p(teo) = .6 ´ 1 ´ .88 ´ .6 ´ .07 ´ .3 ≅ .00665
p(toy) = .6 ´ 1 ´ .88 ´ .3 ´ .07 ´ .1 ≅ .00111
p(tty) = .6 ´ 1 ´ .12 ´ 1 ´ 1 ´ .1 ≅ .0072

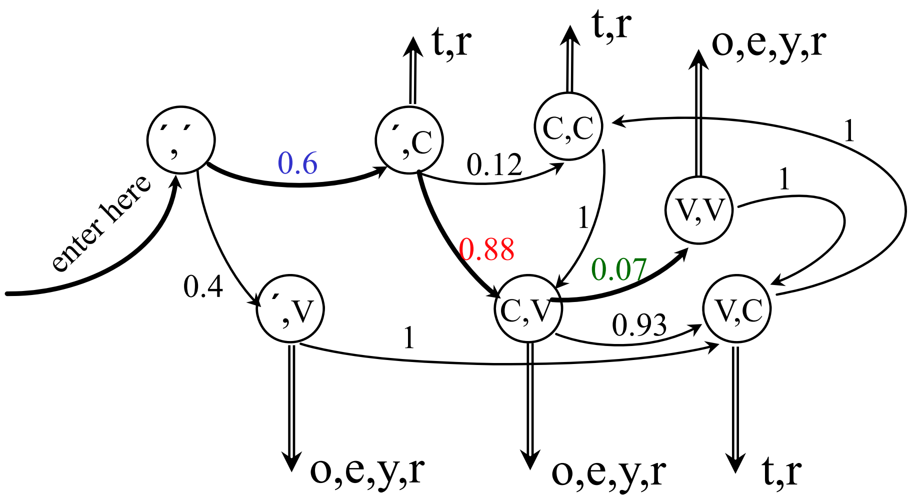
Class Trigrams: the Trellis

- Trellis generation (Y = “toy”):



Overlapping Classes

- Imagine that classes may overlap
  - e.g. ‘r’ is sometimes vowel sometimes consonant, belongs to V as well as C:

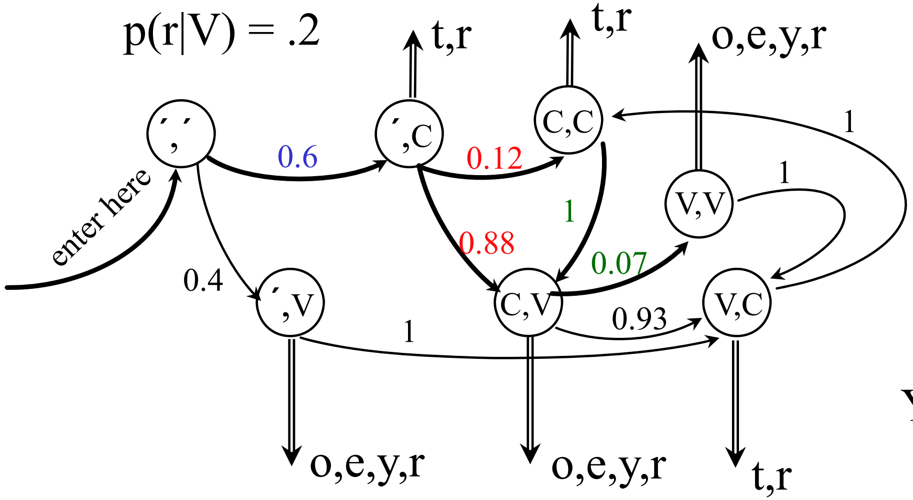


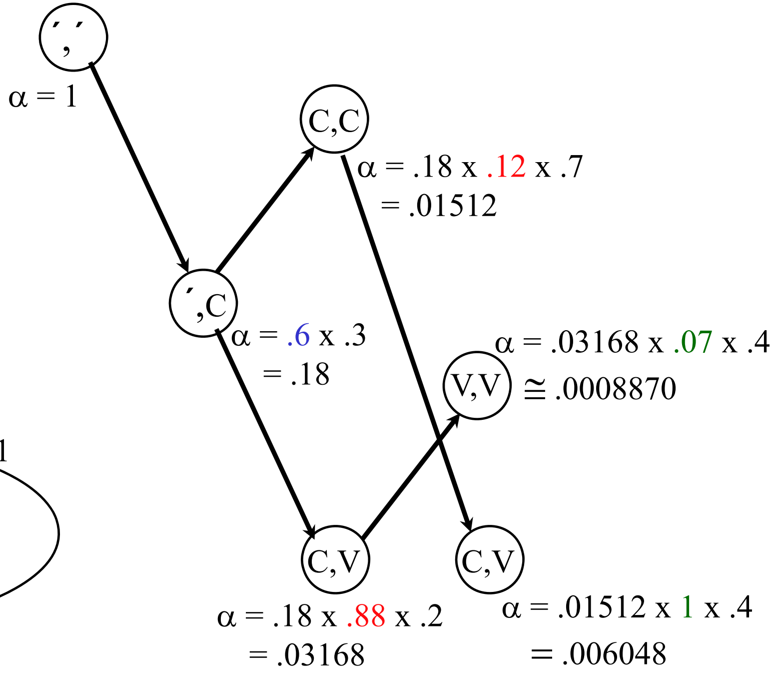
$p(t|C) = .3$
$p(r|C) = .7$
$p(o|V) = .1$
$p(e|V) = .3$
$p(y|V) = .4$
$p(r|V) = .2$

$p(try) = ?$

Overlapping Classes: Trellis Example

p(t|C) = .3
p(r|C) = .7
p(o|V) = .1
p(e|V) = .3
p(y|V) = .4
p(r|V) = .2





Y: t r y p(Y) = .006935

Trellis: Remarks

- So far, we went left to right (computing α)
- Same result: going right to left (computing β)
  - supposed we know where to start (finite data)
- In fact, we might start in the middle going left <u>and</u> right
- Important for parameter estimation

  (Forward-Backward Algortihm alias Baum-Welch)
- Implementation issues:
  - scaling/normalizing probabilities, to avoid too small numbers & addition problems with many transitions

— too small numbers will result to zero in computer.

The Viterbi Algorithm

- Solving the task of finding the most likely sequence of states which generated the observed data
- i.e., finding

→ our input

$$S_{best} = \mathrm{argmax}_S P(S|Y)$$

which is equal to (Y is constant and thus P(Y) is fixed):

$$S_{best} = \mathrm{argmax}_S P(S,Y) =$$

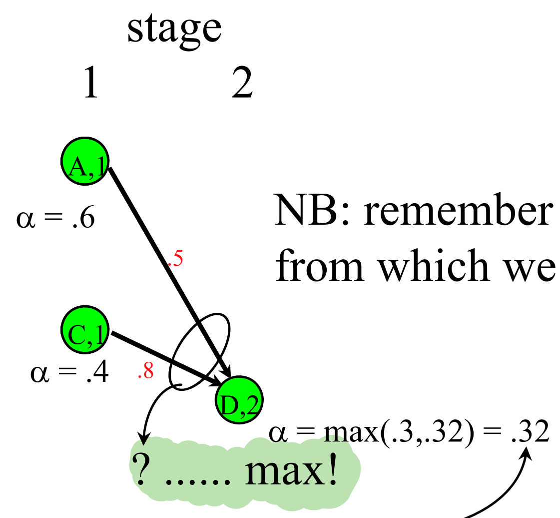
$$= \mathrm{argmax}_S P(s_0,s_1,s_2,\ldots,s_k,y_1,y_2,\ldots,y_k) =$$

$$= \mathrm{argmax}_S \Pi_{i=1..k}\, p(y_i|s_i,s_{i-1})p(s_i|s_{i-1})$$

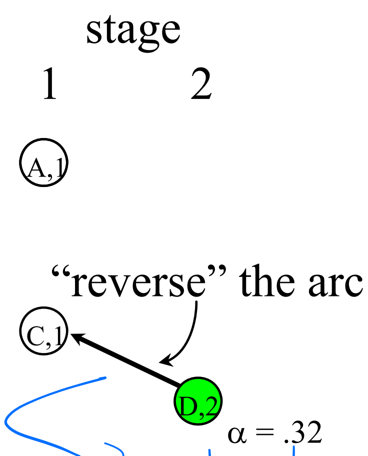
We are not interested in the probability itself, but the sequence

The Crucial Observation

- Imagine the trellis build as before (but do not compute the αs yet; assume they are o.k.); stage i:



NB: remember previous state from which we got the maximum:

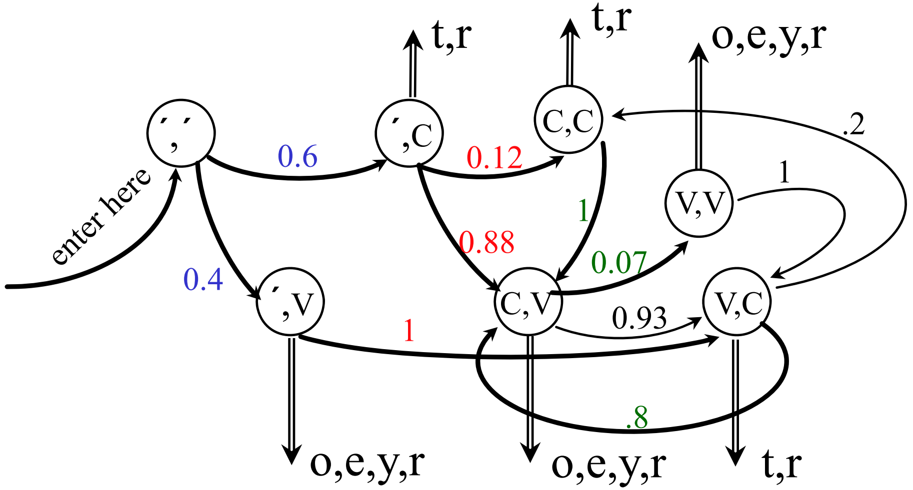


we have to remember from where was the best lambda

this is certainly the "backwards" maximum to (D,2)... but it <u>cannot change even whenever we go forward</u> (M. Property: Limited History)

Viterbi Example

- ‘r’ classification (C or V?, sequence?):



p(t|C) = .3
p(r|C) = .7
p(o|V) = .1
p(e|V) = .3
p(y|V) = .4
p(r|V) = .2

p(XYZ|rry)

argmax_{XYZ} p(~~rry|XYZ~~) = ?

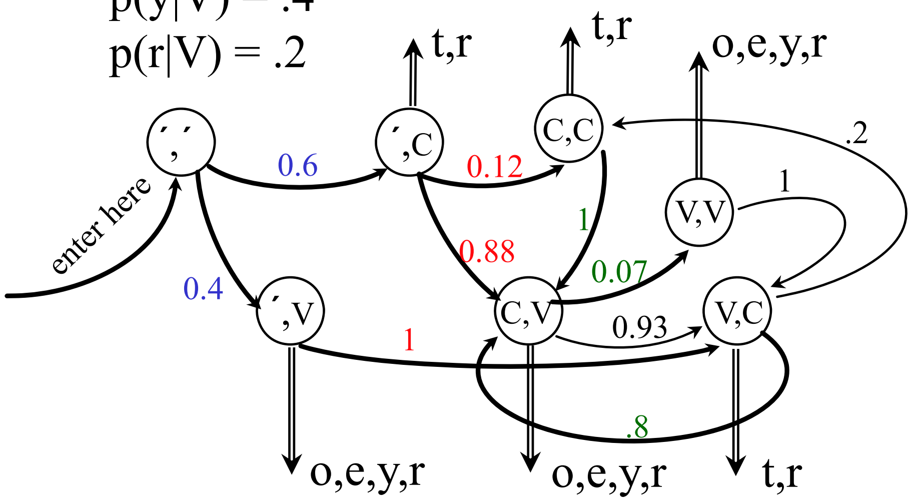
Possible state seq.: (´,v)(v,c)(c,v)[VCV], (´,c)(c,c)(c,v)[CCV], (´,c)(c,v)(v,v) [CVV]

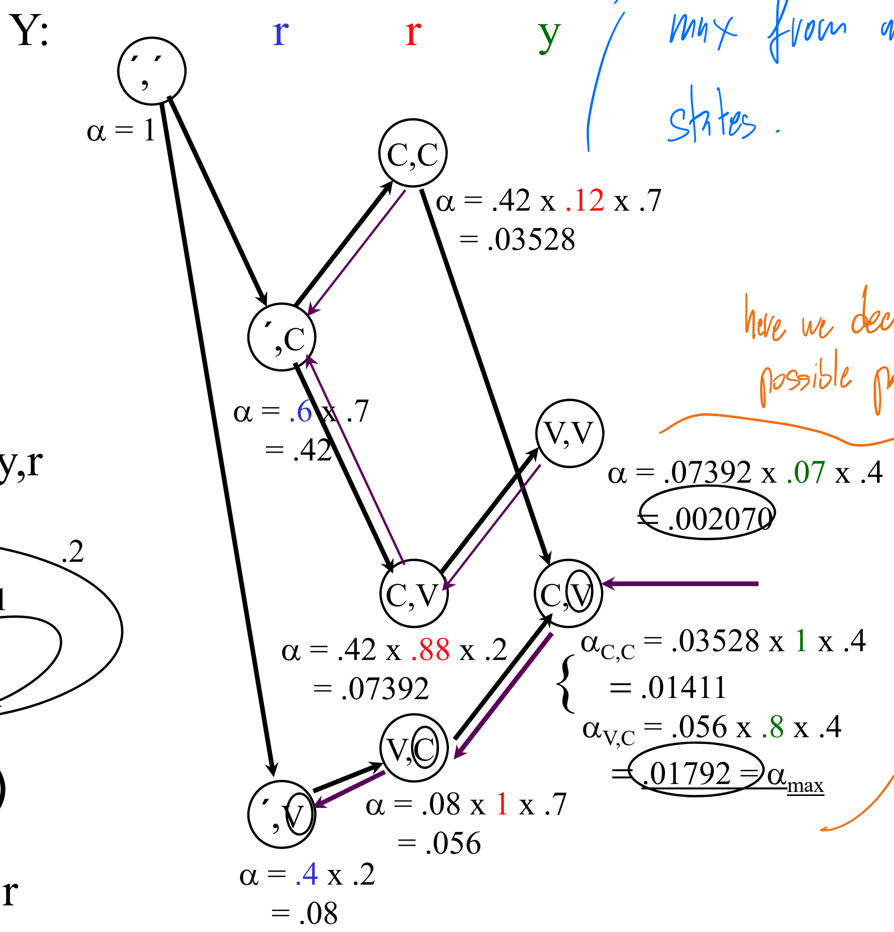
Viterbi Computation

we actually need to take max from all terminating states.

α in trellis state: best prob from start to here

p(t|C) = .3
p(r|C) = .7
p(o|V) = .1
p(e|V) = .3
p(y|V) = .4
p(r|V) = .2

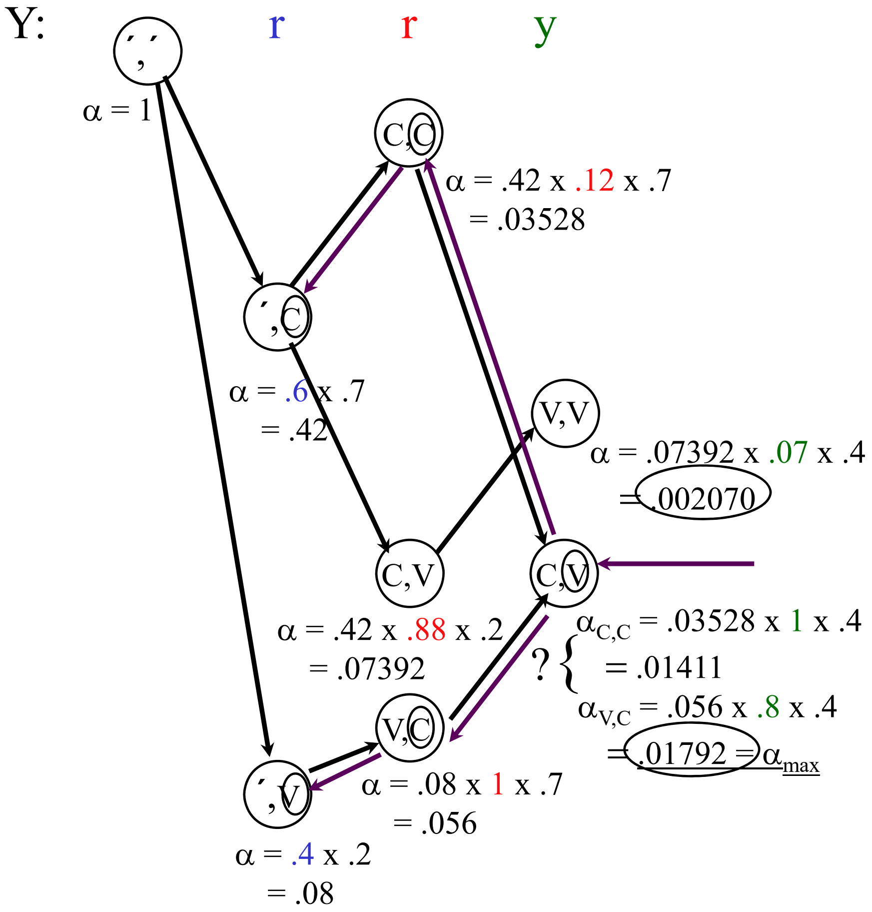




here we decide between two possible paths (in backward)

n-best State Sequences

- Keep track of n best "back pointers":
- Ex.: n= 2: Two "winners": VCV (best) CCV (2nd best)



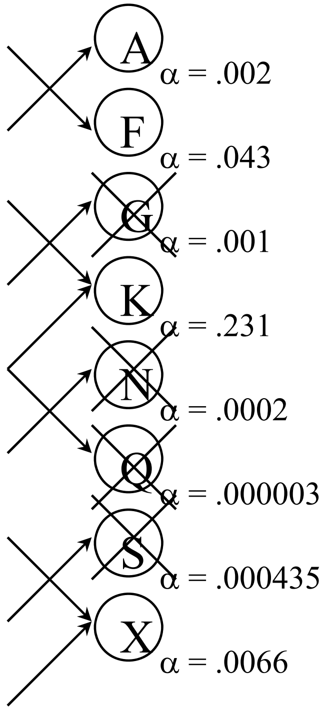
Tracking Back the n-best paths

- Backtracking-style algorithm:
  - **Start at the end, in the best of the n states (s_{best})**
  - **Put the other n-1 best nodes/back pointer pairs on stack, except those leading from s_{best} to the same best-back state.**
- Follow the back "beam" towards the start of the data, spitting out nodes on the way (backwards of course) using always only the best back pointer.
- At every beam split, push the diverging node/back pointer pairs onto the stack (node/beam width is sufficient!).
- When you reach the start of data, close the path, and pop the top-most node/back pointer(width) pair from the stack.
- Repeat until the stack is empty; expand the result tree if necessary.

Those having already very low prob. are very likely not very having high prob. of occurring

Pruning

- Sometimes, too many trellis states in a stage:



This must be tuned well

criteria: (a) α < threshold

(b) Σπ < threshold

(c) # of states > threshold

(get rid of smallest α)

Π := all α that I gonna remove

again, no computation, only how it works or what its good for

HMM Parameter Estimation: the Baum-Welch Algorithm

HMM: The Tasks

- HMM (the general case):
  - five-tuple (S, S_0, Y, P_S, P_Y), where:
    - **$S = \{s_1,s_2,...,s_T\}$ is the set of states, S_0 is the initial state,**
    - **$Y = \{y_1,y_2,...,y_V\}$ is the output alphabet,**
    - **$P_S(s_j|s_i)$ is the set of prob. distributions of transitions,**
    - **$P_Y(y_k|s_i,s_j)$ is the set of output (emission) probability distributions.**
- Given an HMM & an output sequence $Y = \{y_1,y_2,...,y_k\}$:

  ✓(Task 1) compute the probability of Y;

  ✓(Task 2) compute the most likely sequence of states which has generated Y.

  (Task 3) Estimating the parameters (transition/output distributions)

A Variant of EM

- Idea (~ EM, for another variant see LM smoothing):
  - Start with (possibly random) estimates of P_S and P_Y.
  - Compute (fractional) "counts" of state transitions/emissions taken, from P_S and P_Y, given data Y. — we go over the data „and count"
  - Adjust the estimates of P_S and P_Y from these "counts" (using the MLE, i.e. relative frequency as the estimate).
- Remarks: ~ millions of params.
  - many more parameters than the simple four-way smoothing
  - no proofs here; see Jelinek, Chapter 9

Speech recognition is the first usage of this algorithm → we don't have much anotated data for it...

We usually know the structures of HMMs

– we only miss the probs.

Setting

We must give restrictions about the output generation and state transition

→ then EM has where to start with

- HMM (without P_S, P_Y) (S, S_0, Y), and data $T = \{y^i \in Y\}_{i=1..|T|}$
  - **will use T ~ |T|**
- – HMM structure is given: (S, S_0)
- – P_S:Typically, one wants to allow "fully connected" graph
  - **(i.e. no transitions forbidden ~ no transitions set to hard 0)**
  - **why? → we better leave it on the learning phase, based on the data!**
  - **sometimes possible to remove some transitions ahead of time**
- – P_Y: should be restricted (if not, we will not get anywhere!)
  - **restricted ~ hard 0 probabilities of p(y|s,s')** → some situations can never happen
  - **"Dictionary": states ↔ words, "m:n" mapping on S × Y (in general)**

You get small part of text annotated by people, which will serve for MLE init.

Initialization

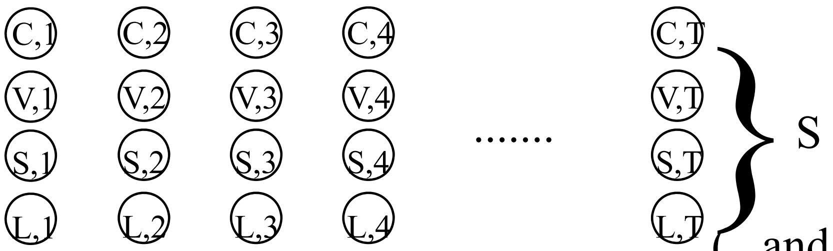
- For computing the initial expected "counts"
- Important part
  - EM guaranteed to find a *local* maximum only (albeit a good one in most cases)
- P_Y initialization more important
  - fortunately, often easy to determine
    - **together with dictionary ↔ vocabulary mapping, get counts, then MLE**
- P_S initialization less important
  - e.g. uniform distribution for each p(.|s)

Data Structures

- Will need storage for:
  - The predetermined structure of the HMM

    (unless fully connected → need not to keep it!)
  - The parameters to be estimated (P_S, P_Y)
  - The expected counts (same size as P_S, P_Y)
  - The training data $T = \{y^i \in Y\}_{i=1..T}$
  - The trellis (if f.c.): ↑ T Size: T ´ S (Precisely, |T|´|S|)

Each trellis state:
two [float] numbers
(forward/backward)



(...and then some)

The Algorithm Part I

1. Initialize P_S, P_Y
2. Compute "forward" probabilities:
   - **follow the procedure for trellis (summing), compute $\alpha(s,i)$**
   - **use the current values of P_S, P_Y ($p(s'|s)$, $p(y|s,s')$):**

$$\alpha(s',i) = \sum_{s \to s'} \alpha(s,i-1) \times p(s'|s) \times p(y_i|s,s')$$

→ as if computing the prob. of the word in the input.

   - **NB: do not throw away the previous stage!**
3. Compute "backward" probabilities
   - **start at all nodes of the last stage, proceed backwards, $\beta(s,i)$**
   - **i.e., probability of the "tail" of data from stage *i* to the end of data**

„tail prob."

$$\beta(s',i) = \sum_{s \leftarrow s'} \beta(s,i+1) \times p(s|s') \times p(y_{i+1}|s',s)$$

→ prob. of the tail of the data given some state

   - **also, keep the $\beta(s,i)$ at all trellis states**

How to determine the number of states? ~> simple: # classes ~ # states

The Algorithm Part II

4. Collect counts:
   - for each output/transition pair compute

$$c(y,s,s') = \Sigma_{i=0..k-1, y=y_{i+1}} \alpha(s,i)\ p(s'|s)\ p(y_{i+1}|s,s')\ \beta(s',i+1)$$

one pass through data, only stop at (output) y

prefix prob.

this transition prob ´ output prob

tail prob

$$c(s,s') = \Sigma_{y \in Y} c(y,s,s') \text{ (assuming all observed } y_i \text{ in Y)}$$

$$c(s) = \Sigma_{s' \in S} c(s,s')$$

5. Reestimate: $p'(s'|s) = c(s,s')/c(s)$ $p'(y|s,s') = c(y,s,s')/c(s,s')$
6. Repeat 2-5 until desired convergence limit is reached.

Baum-Welch: Tips & Tricks

- Normalization badly needed
  - long training data → extremely small probabilities
- Normalize α,β using the same norm. factor:

$$N(i) = \Sigma_{s \in S}\ \alpha(s,i)$$

→ must be the same factor for both α and β

as follows:

- **compute α(s,i) as usual (Step 2 of the algorithm), computing the sum N(i) at the given stage *i* as you go.**
- **at the end of each stage, recompute all αs (for each state s):**

□ $$\alpha^*(s,i) = \alpha(s,i) / N(i)$$

it will therefore be pseudonormalization

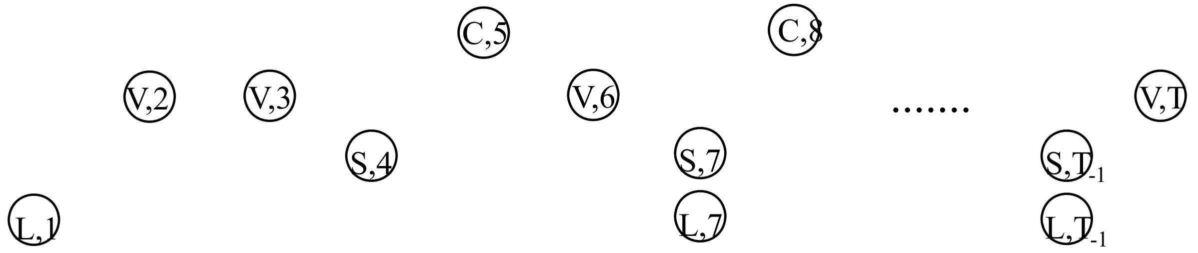
- **use the same N(i) for βs at the end of each backward (Step 3) stage:**

□ $$\beta^*(s,i) = \beta(s,i) / N(i)$$

Example

- Task: pronunciation of “the”
- Solution: build HMM, fully connected, 4 states:
  - **S - short article, L - long article, C,V - starting w/consonant, vowel**
  - **thus, only “the” is ambiguous (a, an, the - not members of C,V)**
- Output from states only (p(w|s,s’) = p(w|s’))
- Data Y: an egg and a piece of the big .... the end

Trellis:



Example: Initialization

- Output probabilities:

  $p_{init}(w|c) = c(c,w) / c(c)$; where $c(S,the) = c(L,the) = c(the)/2$

  (other than that, everything is deterministic)

- Transition probabilities:
  - $p_{init}(c'|c) = 1/4$ (uniform)
- Don't forget:
  - about the space needed
  - initialize $\alpha(X,0) = 1$ (X : the never-occurring front buffer st.)
  - initialize $\beta(s,T) = 1$ for all s (except for s = X)

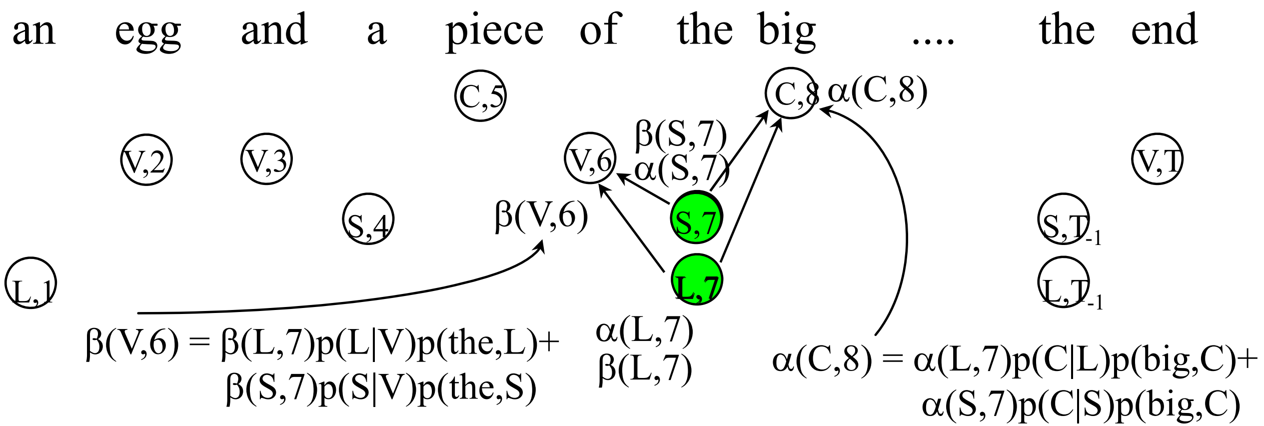
Fill in alpha, beta

- Left to right, alpha:

$$\alpha(s',i) = \sum_{s \to s'} \alpha(s,i-1) \times p(s'|s) \times p(w_i|s')$$

output from states

- Remember normalization (N(i)).
- Similarly, beta (on the way back from the end).

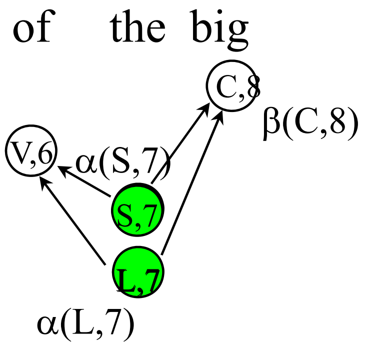


Counts & Reestimation

- One pass through data
- At each position *i*, go through all pairs (s_i, s_{i+1})
- Increment appropriate counters by frac. counts (Step 4):
  - **$inc(y_{i+1}, s_i, s_{i+1}) = a(s_i, i)\ p(s_{i+1}|s_i)\ p(y_{i+1}|s_{i+1})\ b(s_{i+1}, i+1)$**
  - **$c(y, s_i, s_{i+1})$ += inc (for y at pos i+1)**
  - **$c(s_i, s_{i+1})$ += inc (always)**
  - **$c(s_i)$ += inc (always)**

inc(big,L,C) = $\alpha(L,7)p(C|L)p(big,C)\beta(C,8)$

inc(big,S,C) = $\alpha(S,7)p(C|S)p(big,C)\beta(C,8)$



- Reestimate p(s'|s), p(y|s)
  - **and hope for increase in p(C|S) and p(V|L)...!!**

HMM: Final Remarks

- Parameter "tying":
  - keep certain parameters same (~ just one "counter" for all of them)
  - any combination in principle possible
  - ex.: smoothing (just one set of lambdas)
- Real Numbers Output
  - Y of infinite size (R, R^n):
    - **parametric (typically: few) distribution needed (e.g., "Gaussian")**
- "Empty" transitions: do not generate output
    - **~ vertical arcs in trellis; do not use in "counting"**